Paraproducts and Analysis of Rough Path

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Organizers:

Massimiliano Gubinelli
Universität Bonn, Germany

Christoph Thiele
Universität Bonn, Germany

[*]supported by Hausdorff Center for Mathematics, Bonn

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1 Controlling rough paths (part II)

after M. Gubinelli [1]
A summary written by Gianmarco Brocchi

Abstract

We study the problem of existence, uniqueness and continuity of solutions of differential equations driven by irregular paths with Hölder exponent greater than $\frac{1}{3}$ (e.g. samples of Brownian motion). We will also show how this setting relates to known stochastic integrals.

1.1 Introduction

Consider an interval $J \subseteq \mathbb{R}$ and a γ-Hölder path X in $\mathcal{C}^\gamma(J, V)$ taking values in a finite dimensional vector space V. Let φ be a function in $C(V, V \otimes V^\star)$. We are interested in studying the controlled differential equation

$$\boxed{dY_t^\mu = \varphi(Y_t)^\mu_\nu \, dX_t^\nu} \quad Y_{t_0} = y, \; t_0 \in J \tag{1}$$

where μ, ν are vector indices[1]. A solution to (1) is, formally, a continuous path $Y \in \mathcal{C}^\gamma(J, V)$ such that

$$Y_t^\mu = y + \int_{t_0}^t \varphi(Y_u)^\mu_\nu \, dX_u^\nu \tag{2}$$

for every $t \in J$. When $\gamma > \frac{1}{2}$ appropriate conditions on φ allow one to consider the integral in (2) as a Young integral. When $\frac{1}{2} \geq \gamma > \frac{1}{3}$ the integral must be understood as integral of a weakly-controlled path, as in part I of [1].
In the latter case, given a rough path $(X, \mathbb{X}^2)$, the solution of the differential equation (1) is a weakly-controlled path in $\mathcal{D}_X^{\gamma, 2\gamma}(J, V)$.

To prove these results we show that the solution map

$$Y \mapsto G(Y)_t = Y_{t_0} + \int_{t_0}^t \varphi(Y_u)^\mu_\nu \, dX_u^\nu \tag{3}$$

is locally a strict contraction on a subset of the Banach space $\mathcal{C}^\gamma(J, V)$ of Hölder continuous functions on J with values in a finite vector space

[1] We will use Einstein notation omitting summation over repeated indices.

V. Therefore (3) has an unique fixed-point. Moreover, the Itô map $Y = F(y, \varphi, X)$ which sends the data of the differential equation to the solution is Lipschitz continuous (on compact intervals J) in each argument.

The following table summarizes the sufficient hypotheses and our main results in the two cases. The parameter δ is assumed to be in $(0, 1)$.

	$\gamma > 1/2$	$1/2 \geq \gamma > 1/3$
Integral in (2)	Young integral	Integral based on $(X, \mathbb{X}^2)$
Solution	$Y \in \mathcal{C}^\gamma$	$Y \in \mathcal{D}_X^{\gamma,2\gamma}$
Conditions for Existence	$\varphi \in C^\delta(V, V \otimes V^\star)$, $(1+\delta)\gamma > 1$	$\varphi \in C^\delta(V, V)$, $(2+\delta)\gamma > 1$
Stronger condition for Uniqueness	$\varphi \in C^{1,\delta}(V, V \otimes V^\star)$	$\varphi \in C^{2,\delta}(V, V)$

1.2 Preliminaries

We indicate with $\Omega\mathcal{C}$ the set of bounded functions on $\mathbb{R}^2$. For this set of functions we can introduce the norm:

$$\|A\|_\gamma := \sup_{s,t \in \mathbb{R}^2} \frac{|A_{st}|}{|t-s|^\gamma}.$$

The space $\Omega\mathcal{C}^\gamma$ is the subspace of $\Omega\mathcal{C}$ such that $\|A\|_\gamma < \infty$.
For a path X on $I \subset \mathbb{R}$, the map $(\delta X)_{st} := X_t - X_s$ maps $\mathcal{C}^\gamma$ to $\Omega\mathcal{C}^\gamma$.

Lemma 1. *Let $I = [a, b]$ and $\gamma, \eta \in \mathbb{R}$. If $\gamma < \eta$ then*

$$\|\cdot\|_{\gamma,I} \leq |b-a|^{\eta-\gamma} \|\cdot\|_{\eta,I}$$

i.e. the inclusion $\Omega\mathcal{C}^\eta(I) \hookrightarrow \Omega\mathcal{C}^\gamma(I)$ is continuous.

Lemma 2. *Let I, J be two adjacent intervals on $\mathbb{R}$ and let X be a path in $\mathcal{C}^\gamma(I, V)$ and in $\mathcal{C}^\gamma(J, V)$. If $NX \in \mathcal{C}^{\gamma_1,\gamma_2}(I \cup J, V)$, with $\gamma_1 + \gamma_2 = \gamma$, then*

$$\|X\|_{\gamma, I\cup J} \leq 2(\|X\|_{\gamma,I} + \|X\|_{\gamma,J}) + \|NX\|_{\gamma_1,\gamma_2,I\cup J}$$

and $X \in \mathcal{C}^\gamma(I \cup J, V)$.

1.3 Existence and uniqueness when $\gamma > \frac{1}{2}$

Proposition 3 (Existence)**.** *If $\gamma > 1/2$ and $\varphi \in C^\delta(V, V \otimes V^\star)$, $\delta \in (0,1)$ with $(1+\delta)\gamma > 1$, there exists a path $Y \in \mathcal{C}^\gamma(J,V)$ that is a solution of the differential equation* (1)*. (The integral in* (2) *must be understood as Young integral.)*

Sketch of the proof. Start with an interval $I = [t_0, t_0+T] \subseteq J$ for $T > 0$. Under the condition $(1+\delta)\gamma > 1$, G maps $\mathcal{C}^\gamma(I,V)$ to itself. Using decomposition of path in $\mathcal{C}^\gamma$ we can fix a compact, convex subset Q_I which is invariant under G. It can be shown that the map G is continuous on Q_I so, by the Leray-Schauder-Tychnoff theorem, there exists a fixed-point for G in Q_I. We conclude by covering J with a collection of intervals I of small length and patching together local solutions using Lemma 2. □

Proposition 4 (Uniqueness)**.** *If $\gamma > 1/2$, $\varphi \in C^{1,\delta}(V, V \otimes V^\star)$, $\delta \in (0,1)$ with $(1+\delta)\gamma > 1$, there exists a* unique *solution Y in $\mathcal{C}^\gamma(J,V)$ of the differential equation* (1)*. The Itô map $F(y,\varphi,X)$ is Lipschitz continuous in the following sense:*

$$\|F(y,\varphi,X) - F(\widetilde{y},\widetilde{\varphi},\widetilde{X})\|_{\gamma,J} \leq M(\|X-\widetilde{X}\|_{\gamma,J} + \|\varphi-\widetilde{\varphi}\|_{1,\delta} + |y-\widetilde{y}|)$$

with a constant M depending only on $\|X\|_{\gamma,J}$, $\|\widetilde{X}\|_{\gamma,J}$, $\|\varphi\|_{1,\delta}$, $\|\widetilde{\varphi}\|_{1,\delta}$ and J.

Idea of the proof. For $T < 1$ we can fix an invariant compact set Q_I as in the previous Proposition. For T small enough G can be shown to be locally a strict contraction on Q_I, this means we can take $\alpha = \alpha(T) < 1$ such that

$$\|G(Y) - G(\widetilde{Y})\|_{\gamma,I} \leq \alpha\|Y - \widetilde{Y}\|_{\gamma,I}$$

when $Y, \widetilde{Y} \in Q_I$ and $X = \widetilde{X}$. As a strict contraction G has a unique fixed-point on Q_I. Uniqueness for the whole J follows by a covering argument. □

1.4 Existence and uniqueness when $\frac{1}{2} \geq \gamma > \frac{1}{3}$

Proposition 5 (Existence)**.** *If $\gamma > 1/3$, $\varphi \in C^{1,\delta}(V,V)$, $\delta \in (0,1)$ with $(2+\delta)\gamma > 1$, there exists a weakly-controlled path Y in $\mathcal{D}_X^{\gamma,2\gamma}(J,V)$ solution of the differential equation* (1)*. (The integral in* (2) *must be understood as based on the pair $(X, \mathbb{X}^2)$.)*

Sketch of the proof. The path $\varphi(Y)$ belongs to $\mathcal{D}_X^{\gamma,(1+\delta)\gamma}(J,V)$,. Integration against X makes sense for $(2+\delta)\gamma > 1$. We claim, similarly to Proposition 3, that G maps $\mathcal{D}_X^{\gamma,2\gamma}(I,V)$ to itself. Using the following decomposition for $Z = G(Y)$;

$$\delta Z^\mu = Z'_\nu \delta X^\nu + R_Z^\mu = \varphi(Y)^\mu_\nu \delta X^\nu + \partial^\kappa \varphi(Y)^\mu_\nu Y'^\kappa_\rho \mathbb{X}^{2,\nu\rho} + Q_Z^\mu$$

we can bound the norm $\|Z\|_{*,I} = \|Z\|_{\mathcal{D}_X(\gamma,2\gamma,I)}$. One can fix a time $T_* < 1$ such that for all $T < T_*$ the set Q'_I is invariant under G.
Then there exists a solution in $\mathcal{D}_X^{\gamma,2\gamma}(I,V)$ for any $I \subseteq J$ small enough. Consider a covering of J with suitable intervals $I_1, \dots, I_n$; patching together local solutions we get a global one $\overline{Y}$ defined on $\cup_i I_i = J$. Again, one can use Lemma 2 iteratively to prove that $\overline{Y}$ belongs to $\mathcal{D}_X^{\gamma,2\gamma}(J,V)$. □

Proposition 6 (Uniqueness). *If $\gamma > 1/3$, $\varphi \in C^{2,\delta}(V,V)$, $\delta \in (0,1)$ with $(2+\delta)\gamma > 1$, there exists a* unique *solution $Y \in \mathcal{D}_X^{\gamma,2\gamma}(J,V)$ of the differential equation* (1)*, where the integral in* (2) *is based on the couple $(X, \mathbb{X}^2)$. Moreover, the Itô map $F(y,\varphi,X,\mathbb{X}^2)$ is Lipschitz continuous.*

Idea of the proof. As in Proposition 4 we decompose J in smaller intervals in order to bound

$$\epsilon_{Z,I} = \|\varphi(Y) - \widetilde{\varphi}(\widetilde{Y})\|_{\infty,I} + \|\varphi(Y) - \widetilde{\varphi}(\widetilde{Y})\|_{\gamma,I} + \|R_Z - R_{\widetilde{Z}}\|_{2\gamma,I}.$$

For I small enough we can find an invariant domain for G. Choosing $T < 1$ small enough there exists an $\alpha < 1$ such that

$$\|G(Y) - G(\widetilde{Y})\|_{*,I} \leq \alpha \, \|Y - \widetilde{Y}\|_{*,I}.$$

Thus G is a strict contraction in $\mathcal{D}_X^{\gamma,2\gamma}(I,V)$ and has a unique fixed-point. Patching together local solutions we get a global one defined on J, that belongs to $\mathcal{D}_X^{\gamma,2\gamma}(J,V)$. □

1.5 Some probability

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X be a standard Brownian motion defined on it taking values in $V = \mathbb{R}^n$. For a fixed $\gamma < \frac{1}{2}$ and a bounded interval I, the process X is almost surely locally γ-Hölder continuous, thus we can choose a version of X in $\mathcal{C}^\gamma(I,V)$.

Via stochastic integration we can define

$$W^{\mu\nu}_{\text{Itô},st} := \int_s^t (X^\mu_u - X^\mu_s) \, \hat{d}X^\nu_u$$

in the sense of the Itô integral (indicated by the hat in $\hat{d}X_u^\nu$) with respect to the filtration $\mathcal{F}_t = \sigma(X_s\,;\, s \le t)$. For any $s, u, t \in \mathbb{R}$ we have:

$$W^{\mu\nu}_{\text{Itô},st} - W^{\mu\nu}_{\text{Itô},su} - W^{\mu\nu}_{\text{Itô},ut} = (X^\mu_u - X^\mu_s)(X^\nu_t - X^\nu_u) \tag{4}$$

and thus we can consider a continuous version $\mathbb{X}^2_{\text{Itô}}$ of the process $(s,t) \mapsto W_{\text{Itô},st}$ such that (4) holds almost surely for all $s, u, t \in \mathbb{R}$.
Using a variation of an argument introduced in [2],[3] to control Hölder-like seminorms of continuous stochastic processes with a corresponding integral norm, it is possible to show that $\mathbb{X}^2_{\text{Itô}}$ belongs to $\Omega\mathcal{C}^{2\gamma}(I, V \otimes V)$.

On the other hand, the Stratonovich integral is given by

$$\mathbb{X}^{2,\mu\nu}_{\text{Strat.},st} := \int_s^t (X^\mu_u - X^\mu_s) \circ \hat{d}X^\nu_u.$$

From stochastic integration we know that

$$\mathbb{X}^{2,\mu\nu}_{\text{Strat.},st} = \mathbb{X}^{2,\mu\nu}_{\text{Itô},st} + \frac{g^{\mu\nu}}{2}(t-s)\,, \quad \text{where } g^{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Also in this case we can choose a continuous version of $\mathbb{X}^2_{\text{Strat.},st}$ in $\Omega\mathcal{C}^{2\gamma}$ such that (4) holds.

We have introduced the following integrals so far:

Itô integral $\delta I^\mu_{\text{Itô},st} = \int_s^t \varphi(X_u)^\mu_\nu \, \hat{d}X^\nu_u$	Integral based on $(X, \mathbb{X}^2_{\text{Itô}})$ $\delta I^\mu_{\text{rough},st} = \int_s^t \varphi(X_u)^\mu_\nu \, dX^\nu_u$
Stratonovich integral $\delta I^\mu_{\text{Strat.},st} = \int_s^t \varphi(X_u)^\mu_\nu \circ \hat{d}X^\nu_u$	Integral based on $(X, \mathbb{X}^2_{\text{Strat.}})$ $\delta J^\mu_{st} = \int_s^t \varphi(X_u)^\mu_\nu \, dX^\nu_u$

The connection between them is pointed out by following theorem:

Theorem 7. *Let $\varphi \in C^{1,\delta}(V, V \otimes V^\star)$ with $(1+\delta)\gamma > 1$ and $\gamma < \frac{1}{2}$. Then each stochastic integral in the left column of the table has a continuous version which almost surely coincides with the integral on the right. Moreover, by the relationship between Itô and Stratonovich integration:*

$$\delta I^\mu_{\textit{Itô}} + \frac{g^{\kappa\nu}}{2}\int_s^t \partial_\kappa \varphi(X_u)^\mu_\nu \, du = \delta I^\mu_{\textit{Strat.}}\,,$$

we have that

$$\delta I^{\mu}_{rough,st} + \frac{g^{\kappa\nu}}{2} \int_s^t \partial_\kappa \varphi(X_u)^{\mu}_{\nu}\, du = \delta J^{\mu}_{st}.$$

References

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Gianmarco Brocchi, University of Bonn
email: **gianmarcobrocchi@gmail.com**