Concordances in 3-manifolds

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Introduction

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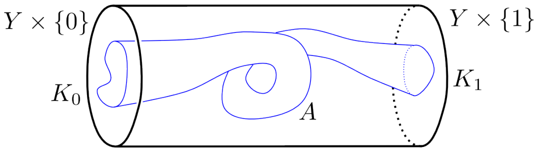

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If $K_0 \sim K_1$, then $[K_0] = [K_1]$, so we have the splitting:

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If $Y = S^3$, the connected sum endows $\mathcal{C} = \mathcal{C}^{S^3}$ with a **group** structure. Otherwise there is no fancy algebra in $\mathcal{C}^Y$.

However there is a natural action $\mathcal{K}(S^3) \curvearrowright \mathcal{C}^Y$ given by:

$$(S^3, K) \cdot [(Y, K')] = [(Y, K\#K')]$$

Definition

Two knots $K_0, K_1 \in \mathcal{K}(Y)$ are **almost-concordant**, $K_0 \dot{\sim} K_1$, if there exist two knots $K_0', K_1' \in \mathcal{K}(S^3)$ such that

$$K_0\#K_0' \sim K_1\#K_1'$$

Concordant knots are also almost-concordant. The converse is already false for S^3!

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$K \in \mathcal{K}(Y)$ can be:

- **Nullhomologous** if it represents the class $0 \in H_1(Y;\mathbb{Z})$.
  $\Leftrightarrow$ boundary of embedded surfaces in Y.
- **Local** if it is contained in a 3-disk embedded in Y.
  $\Leftrightarrow (Y,K) = (S^3, K')\#(Y, \bigcirc)$.
- **Prime** if $(Y,K) = (Y,K_0)\#(S^3,K_1) \Rightarrow K_1 = \bigcirc$.

Note that the only relation is **local** $\Longrightarrow$ **nullhomologous**.

Theorem (Kirby-Lickorish for S^3):

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A "concrete" example

Lens spaces

Closed 3–manifolds $L(p, q)$ obtained by $-\frac{p}{q}$ Dehn surgery on $\bigcirc \subset S^3$.

Using a modified version of Ozsváth-Szabó/Rasmussen's τ-invariant, we can obstruct the existence of almost-concordances between knots in $L(p, q)$.

In S^3 it is an homomorphism:

$$\tau : \mathcal{C} \longrightarrow \mathbb{Z}$$

Extracted from the filtered quasi-isomorphism type of the knot Floer complex $\widehat{CFK}(S^3, K)$.

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In $L(p,q)$:

$$\tau = (\tau^0, \dots, \tau^{p-1}) : \mathcal{C}^{L(p,q)} \longrightarrow \mathbb{Z}^p$$

Behaves in a controlled way under the action of $\mathcal{C}$: if $(L(p,q), K) = (S^3, K_0) \# (L(p,q), K_1)$

$$\tau^i(K) = \tau(K_0) + \tau^i(K_1)$$

Hence we can define the τ-**shifted invariant**

$$\tau_{sh}(K) = (\tau^0(K) + n, \dots, \tau^{p-1}(K) + n)$$

where n is the only integer such that $\min_i \tau^i(K) = 0$.

τ_{sh} is an almost-concordance invariant for knots in lens spaces.

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Proposition

If $K \in \mathcal{K}(L(p,q))$ is local, then $\tau_{sh}(K) = (0, \dots, 0)$.

Then use Baker-Grigsby-Hedden's combinatorial reformulation of $\widehat{HFK}(L(p,q), K)$, compute a couple examples and find: $\widetilde{K} \in \mathcal{K}(L(3,1))$ such that $[\widetilde{K}] = 0$ and

$$\tau(\widetilde{K}) = \tau_{sh}(\widetilde{K}) = (\tau^0(\widetilde{K}), \tau^1(\widetilde{K}), \tau^2(\widetilde{K})) = (1, 0, 0)$$

$\widetilde{K}$ is not almost-concordant to the unknot in $L(3,1)$!
More generally $\widetilde{K}$ is not even almost-concordant to any local knot in $L(3,1)$ (and it is prime).

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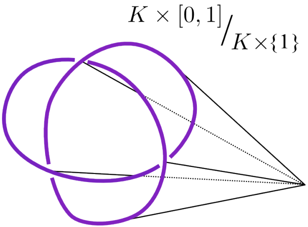
A PL surface is a properly embedded surface in a 4-manifold, smooth everywhere except a finite number of singular points, which are cones over knots.



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(A. Levine) Almost-concordance is equivalent to PL-concordance. Hence τ_{sh} obstructs the existence of PL-concordances!

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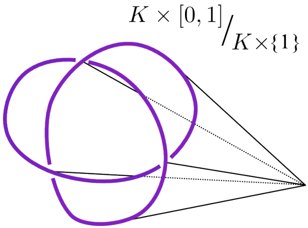

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