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The Hausdorff dimension of the boundary of the Mandelbrot set

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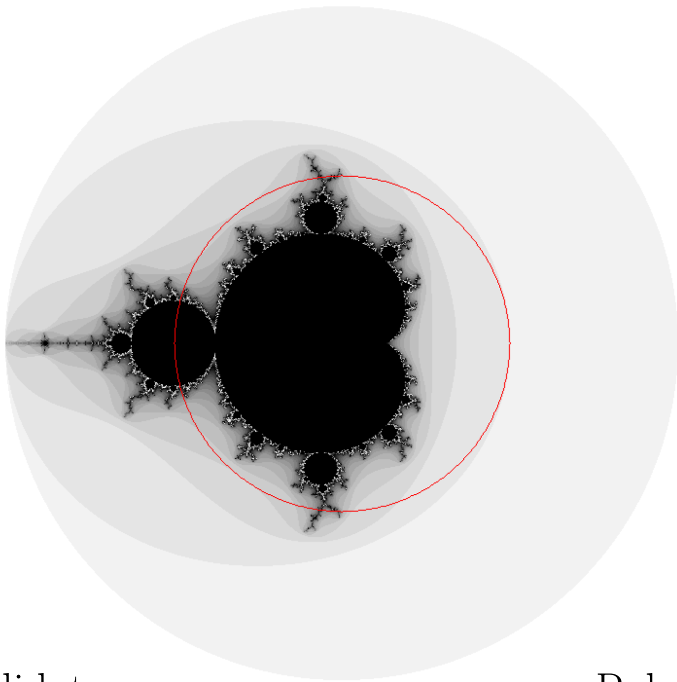

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Foreword

Fractals The so-called "fractal" figures are probably one of the mathematical topic most known among the general public; on one hand they usually show a structure that is immediately pleasant to the observer, but on the other the eye cannot capture entirely the rule that defines the image. Even though there is no specific definition of what a fractal is, the feaures that are mostly considered peculiarity of fractals are

- the self-similarity: there are parts of the figure that look, in some way, "similar" to the whole shape; and
- the infinite complexity: regardless of how much we can zoom inside the figure, its edges appear to be extremely jagged and random.

While the Mandelbrot set exhibits both of these behaviours, we will address our attention exclusively to the second one, that is also far more evident than the first one to the casual observer.

The Mandelbrot set The Mandelbrot set was described for the first time in a slightly different form by Brooks and Matelski in 1978 [BM] and then by Mandelbrot in 1980 [Man] (where the function $z \mapsto \lambda z(1-z)$ was used instead of $z \mapsto z^2 + \lambda$). It appears naturally when considering the structure of the Julia sets (that is, the loci of points that have "chaotic" orbits under repeated iteration of some chosen function) associated to polynomials of degree 2 (that can be put in any of the forms depicted before with an affine change of coordinates). We can think of the Mandelbrot set as a "map" that carries some information about the Julia sets: more precisely, a point λ belongs to the Mandelbrot set whenever the Julia set for the function $z \mapsto z^2 + \lambda$ is connected.

Usually the Julia sets are highly non-regular too: even if it's possible to build Julia sets that are regular manifolds, this is not the case for most of them (moreover, a regular Julia set doesn't imply at all a regular dynamics on it: remember that Julia sets are defined as the points where the dynamics is chaotic). Differently from Julia sets, the Mandelbrot set isn't itself "generated" from dynamic considerations: however there are some structural properties about the complexity of Julia sets that can be brought to the Mandelbrot set. We will exploit one of these results for our main theorem.

Hausdorff dimension The Hausdorff dimension is a generalization of the concept of dimension used on vectorial spaces and manifolds, introduced by Felix Hausdorff in 1918

[Hau]. Intuitively, the dimension of a space expresses the law of scale between the linear dimension of some object and its "measure": that is, we say that a figure has dimension one when, scaling it by a certain factor μ, its measure also scales with the same factor; if it had dimension two, its measure would have scaled with μ^2. Employing such sort of ideas one can define many different generalizations for the notion of dimension, widely diversified depending on the hypoteses needed on the space and the set of the possible values the dimension can take (that is, only integers or also real).

The Hausdorff dimension is one of the most used among these generalizations. It applies to any metric space and gives out a real number (one can actually build spaces that have any Hausdorff dimension in $[0, +\infty]$). Moreover, for a set of dimension d is meaningful to talk about the d-dimensional Hausdorff measure (that coincides, up to a moltiplicative factor, with the external d-dimensional Lebesgue measure when d is an integer and we're mesuring a subset of $\mathbb{R}^d$).

The Hausdorff dimension is usually interesting when dealing with fractal sets, whose dimension frequently is non integral or perhaps integral, but different from what one would expect. For example, the standard Cantor set is known to have Hausdorff dimension $\log 2/\log 3$.

Here we find our main result: while sufficiently regular sets of the plane have boundary with dimension 1, the Mandelbrot set is so irregular that its boundary has Hausdorff dimension not only greater than 1, but even equal to 2 (which is, of course, the maximum a subset of $\mathbb{C}$ can achieve). This result is due to Mitsuiro Shishikura and was first announced in 1991, then published in *Annals of Mathematics* in 1998 [Shi].

To my grandfathers,
Giovanni and Giuseppe

CHAPTER 1

Definitions

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1.1 Dynamical systems

The very fundamental objects in this work are dynamical systems. A dynamical system is the mathematical modelization of some type of system that evolves with time: for example, many different physical, economic or sociologic configurations can be seen as dynamical systems, assuming that some "law" describing how the system changes as time advances is known.

The evolution of a dynamical system can be viewed as a map from the set of all the possible time values to the set of all the possible system configurations. The set of times is usually a (possibly unlimited) interval of real or integer numbers: in the former case the system is *continuous*, in the latter is *discrete*. We will only consider discrete dynamical systems, and particularly those defined on the set $\mathbb{N}$ of natural numbers.

In this case, the above-mentioned "law" that describes the evolution of the system is just a function f from the space of configurations to itself. Thus, if a certain system is in the state x on time t, it will be in the state $f(x)$ at time $t+1$. These considerations lead to our first definitions.

Definition 1.1 (dynamical system)**.** Given some set U (the *space of configurations*), a *discrete dynamical system* is a function $f\colon U \to U$. Since we will discuss only discrete systems, the word "discrete" will be omitted most of the times.

A sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq U$ is called an *orbit* (resp. *reverse orbit*) if, for each $i \in \mathbb{N}$, we have that $f(x_i) = x_{i+1}$ (resp. $f(x_{i+1}) = x_i$).

A set $V \subseteq U$ is called *forward invariant* (resp. *backward invariant*) for f when $f(V) \subseteq V$ (resp. $f^{-1}(V) \subseteq V$).

It is obvious that every point induces exactly one orbit; but each point could have any number (including zero of infinite) of reverse orbits.

Now we can ask ourselves a few questions on how such a system works: for example, do these orbits have some limit points, maybe the same for different orbits? Do these orbit behave regularly, or do they progress in a chaotic fashion? How much does the orbit change when I move slightly the base point? Of course this sort of questions makes no sense unless we add some more structure to the set U: for example, we can take a topological space, a measure space, a differentiable manifold or a complex manifold; and, at the same time, we will ask the function f to be continuous, measurable, C^1 or holomorphic.

In this work we shall concentrate on holomorphic dynamical systems of the Riemann sphere; we then have that f is actually a rational function. It turns out that such a rigid structure enables us to draw out many interesting facts about its dynamical properties.

1.2 The topology of locally uniform convergence

Before going further with our main subject, we have to recall a few classic results about the so-called *locally uniform convergence*, which will be the topology that we will consider on functional spaces. The intersted reader will be able to find proofs for this facts, for instance, in [Mil], §3.

Let S and T be Riemann surfaces (in practice they will always be the Riemann sphere or some its open set): let us indicate with $\mathrm{Hol}(S,T)$ the space of all holomorphic map defined on S with values in T. It is a known fact that each Riemann surface is compact or σ-compact, i.e., can be written as countable union of compact sets $K_1 \subset K_2 \subset \dots$, where each K_i is contained in the interior of the next one. It is then defined a topology over $\mathrm{Hol}(S,T)$ that is induced by the distance:

$$\mu(f,g) = \sum_{n\in\mathbb{N}} \frac{1}{2^n} \min\left\{ 1, \max_{x\in K_n} d(f(x),g(x)) \right\},$$

where d is a distance on the surface T.

Proposition 1.2. $\mathrm{Hol}(S,T)$ *with the function* μ *is actually a metric space. The topology induced does not depend on the particular choice of the sets* $\{K_i\}$*. Moreover, we have that* $f_n \to f$ *if and only if for each compact set* $K \subseteq S$ *we have that* $f_n|_K \to f|_K$ *uniformly. That's why this topology is also known as the* locally uniform convergence*; it is also known as the* compact-open topolgy*.*

Holomorphic functions have excellent stability properties with respect to the uniform convergence. In particular, we can commute the operations of locally uniform limit and derivative.

Proposition 1.3. *Let* $\{f_n\}_{n\in\mathbb{N}} \subset \mathrm{Hol}(S,\mathbb{C})$ *be holomorphic functions that converge locally uniformly to some function* f*. Then* f *itself is holomorphic and* $f_n' \to f'$ *locally uniformly.*

We can also define a concept of locally uniform *divergence*, that will be required to handle non-compact surfaces.

Definition 1.4. We say that a sequence of maps $\{f_n\} \subset \mathrm{Hol}(S,T)$ *diverges locally uniformly* (or also *diverges compactly*) if for any choice of compact sets $K \subseteq S$ and $K' \subseteq T$ we have that $f_n(K) \cap K'$ is definitively empty as $n \to \infty$.

With these notions, we can introduce a definition that will be helpful to establish when a dynamical system has regular or chaotic properties.

Definition 1.5 (normal families)**.** We shall say that a family of functions $\mathcal{F} \subseteq \mathrm{Hol}(S,T)$ is *normal* whenever for each infinite sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ at least one of these two (non mutually exclusive) conditions is satisfied:

- there is a subsequence of $\{f_n\}$ that converges locally uniformly;
- there is a subsequence of $\{f_n\}$ that diverges locally uniformly.

It also will be helpful to use this well-known result to prove that certain family of functions are normal.

Proposition 1.6. *Let S and T be two hyperbolic Riemann surfaces. Then each family of holomorphic maps from S to T is normal.*

Remember also that any open subset of the Riemann sphere that omits at least three points is a hyperbolic Riemann surface.

Proof. These are Lemma 2.5 and Corollary 3.3 in [Mil]. □

1.3 Fatou and Julia sets of the Riemann sphere

As already pointed out, one of the main questions people can ask about a dynamical system is about its chaotic properties. It is well known that even in systems defined by really simple formulae there are many cases of orbits that does not appear to have any regular behaviour at all. While even the task of defining what is to be considered "chaotic" or "regular" is sometimes really challenging, in holomorphic dynamics there is a rather well established convention. The names involved are those of Pierre Fatou and Gaston Julia, two french mathematicians that began the study of this topic during the first two decades of the 20th century.

We can now give one of the most important definitions.

Definition 1.7 (Julia and Fatou sets)**.** Let $f\colon S \to S$ be a dynamical system on some Riemann surface S. We say that a point $z \in S$ is a *Fatou point* when there is a neighbourhood U of z in S such that the family of iterates $\{f^{\circ n}|_U\}_{n\in\mathbb{N}}$ is normal from U to S.

If z is not a Fatou point, then we say that it is a *Julia point*.

We will indicate by $F(f)$ the *Fatou set* (i.e., the set of Fatou points) for f and by $J(f)$ its *Julia set* (i.e., the set of Julia points).

Remark 1.8. Of course, when the Riemann surface that supports the dynamical system is compact (which will be our case most of the times), the second point in Definition 1.5 cannot happen.

Intuitively, the Fatou set is the region where the dynamic does not changes abruptly when moving the intial condition. On the contrary, the Julia set is the region where changing slightly the initial condition can bring to wildly different behaviours of the system.

Let us now specialize this definition to the case that we will be interested into and put the Riemann sphere $S = \hat{\mathbb{C}}$ as support of our system. It is well known that a holomorphic function of the Riemann sphere into itself must be a rational function, i.e., the quotient of two polynomials.

As before, we will not give proofs for the facts listed here. The reader is again invited to check out existing literature to delve into the subject (again, a good reference is [Mil], §4).

Proposition 1.9 (properties of Fatou and Julia sets)**.** *Let $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of the sphere, with degree 2 or more, and let F and J be respectively its Fatou and Julia set. Then:*

- *both sets are* fully invariant*, i.e., $J(f) = f(J(f)) = f^{-1}(J(f))$ and $F(f) = f(F(f)) = f^{-1}(F(f))$;*
- *both sets do not change taking an iterate of f, i.e., for each $k \in \mathbb{N}^*$ we have $J(f) = J(f^{\circ k})$ and $F(f) = F(f^{\circ k})$;*
- *$J(f)$ cannot be vacuous (although $F(f)$ can);*
- *$J(f)$ is* perfect*; i.e., it has no isolated points;*
- *$J(f)$ is either connected or it has uncountably many connected components;*
- *let $z_0 \in J(f)$ be a Julia point; then its preimages*

  $$\left\{ z \in \hat{\mathbb{C}} \;\middle|\; f^{\circ n}(z) = z_0 \text{ for some } n \geqslant 0 \right\}$$

  are dense in J.

Remark 1.10. Effectively drawing the Julia set for a given function (for example, using a computer program) is usually a difficult task. The last assertion in Proposition 1.9 suggests a way to do it by approximation, finding a Julia point and than tracing its preimages (there are some classes of Julia points that can be found analytically). While such an algorithm has a good numerical stability, the number of preimages grows exponentially, making the actual convergence of the rendered approximation to the real Julia set slow.

1.4 The Mandelbrot set

Let us now consider the case where the function f is a polynomial of degree 2 (the simplest, yet far from trivial case). We want to concentrate on whether $J(f)$ is connected or not. Perhaps after an affine change of coordinates (that, of course, does not change the connectivity of the Julia set) f can be put in the form $P_\lambda(z) = z^2 + \lambda$, for $\lambda \in \mathbb{C}$.

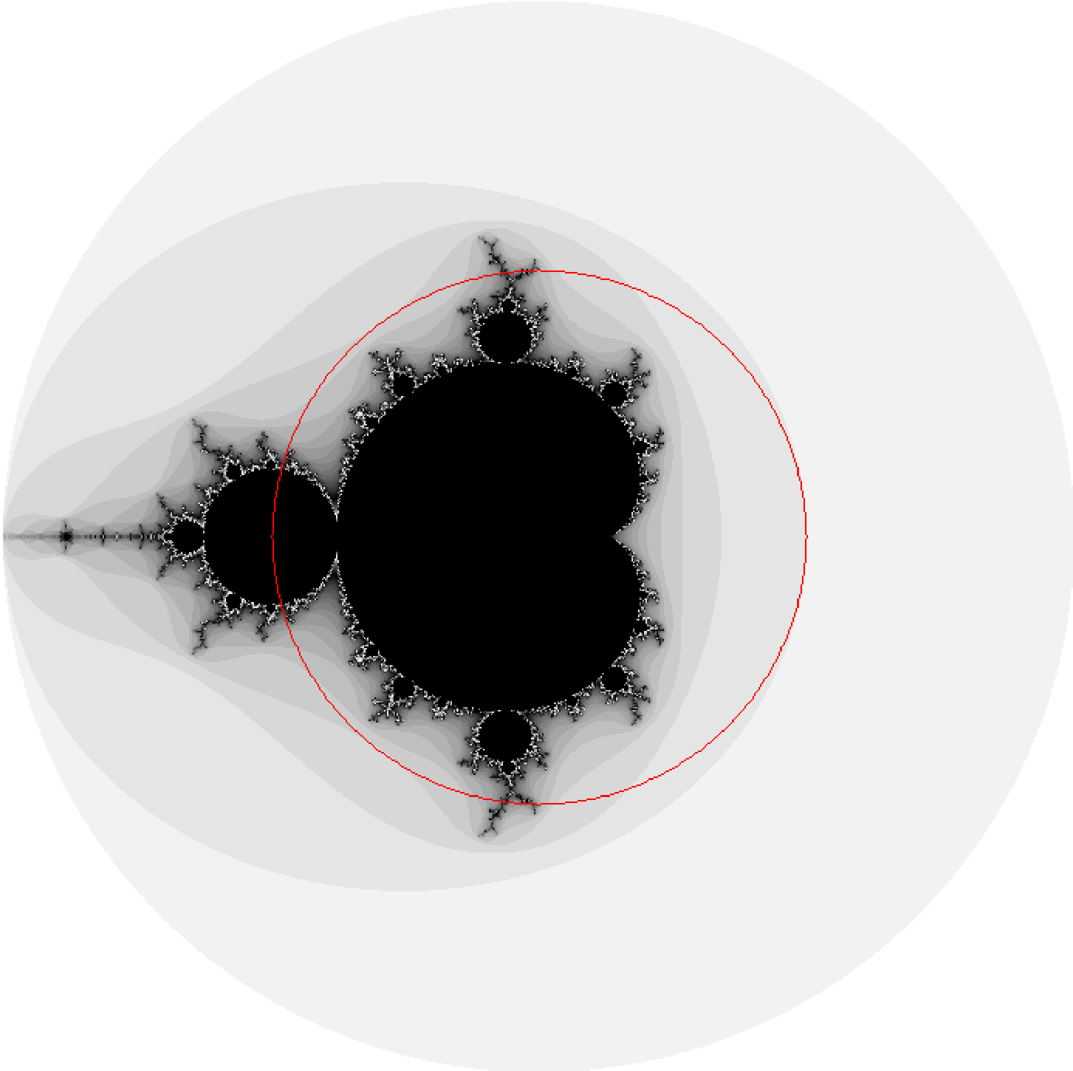

Figure 1.1: Approximations of the Mandelbrot set

Definition 1.11 (Mandelbrot set)**.** The *Mandelbrot set* is the set

$$M := \{ \lambda \in \mathbb{C} \mid J(P_\lambda) \text{ is connected} \} .$$

Of course, without having some more useful characterization, it is really difficult to understand how does this set look like. Fortunately there are a few classical results that strongly help us. Again, the proofs can be found in [Mil], §9.

Definition 1.12 (filled Julia set)**.** Given a rational function f of the sphere, the *filled Julia set* is the set $K(f) \subset \mathbb{C}$ of the complex points whose orbit through f is bounded.

The following two results are then valid. Note that it is fundamental that f is a polynomial.

Proposition 1.13. *For a polynomial f of degree at least 2, the filled Julia set $K(f)$ is compact. Its complement is connected and its topological boundary $\partial K(f)$ is the Julia* *set $J(f)$, while its interior $\mathring{K}(f)$ is the union of all the bounded connected components of the Fatou set $F(f)$.*

Theorem 1.14. *Let f be a polynomial of degree at lest 2. Then the Julia set is connected if and only if the filled Julia set contains all the finite critical points of f.*

In this case, also the filled Julia set is connected. Otherwise, both the Julia and the filled Julia sets have uncountably many components.

Given that the polynomials P_λ have exactly one critical point (the zero), it follows immediately that

$$\begin{aligned} M &= \{ \lambda \in \mathbb{C} \mid \text{the orbit of } 0 \text{ through } P_\lambda \text{ is bounded} \} \\ &= \{ \lambda \in \mathbb{C} \mid \forall n \in \mathbb{N},\ |P_\lambda^{\circ n}(0)| \leqslant 2 \}, \end{aligned} \tag{1.1}$$

where the second equality follows from the (rather easy) fact that an orbit escaping the circle of radius 2 centered in the origin must actually diverge.

We have now a way to draw (rather good) approximations of the Mandelbrot set. We can fix a number of iterations and calculate that number of elements of the orbit casted by any point: if the orbit has at any time modulus greater than 2 then the point is outside the Mandelbrot set. Otherwise, for what we can say, the point is in the Mandelbrot set. Of course, a higher number of iterations gives a better approximation of the set (but also a higher computing time).

Using these insights we can produce images like figure 1.1, where the actual Mandelbrot set is inner black part and the grey tones suggest the number of iterations needed for each point to escape the radius 2 circle. The boundary of the outer grey shape is the radius 2 circle itself, while the red circle is the unit circle.

Proposition 1.15. *Both M and ∂M are compact sets.*

Proof. Both are obviously bounded and ∂M is closed by construction.

It follows from equation (1.1) that the complementary of M is open. Indeed take a point $\lambda \notin M$: then $|P_\lambda^{\circ n}(0)| > 2$ for some sufficiently large n. But the function $\lambda \mapsto P_\lambda^{\circ n}(0)$ is continuous, so there is a whole neighbourhood of λ of points with divergent orbits. □

Remark 1.16. It is worth noting that, for the specific case of the polynomials P_λ, when the Julia set is not connected it is actually totally disconnected. This, together with the topological properties already stated in Proposition 1.9, means that it is homeomorphic to the Cantor set (see [HY]).

1.5 The boundary of the Mandelbrot set as a bifurcation locus

We have defined the Mandelbrot set as the locus of the points for which the corresponding Julia set $J(P_\lambda)$ is connected. That immediately implies that on the boundary of the Mandelbrot set there are infinitely nearby points whose corresponding polynomials express completely different dynamics. In a sense, it happens that actually the converse is true too.

Definition 1.17 (J-stability)**.** Let $\Lambda \subseteq \mathbb{C}$ an open subset and $\{f_\lambda\}_{\lambda\in\Lambda}$ a family of rational maps analytically depending on λ.

We say that $\lambda_0 \in \Lambda$ is J*-stable* if there is some neighbourhood Λ' of λ_0 and a continuous map $h\colon \Lambda' \times J(f_{\lambda_0})$ such that:

- $h_\lambda = h(\lambda, \cdot)$ conjugates $(J(f_{\lambda_0}), f_{\lambda_0})$ with $(J(f_\lambda), f_\lambda)$;
- h_{λ_0} is the identity.

We also say that λ_0 is J*-unstable* if it is not J-stable.

Proposition 1.18. *The boundary of the Mandelbrot set coincides with the* J*-unstable points in the familiy* $\{P_\lambda\}_{\Lambda\in\mathbb{C}}$.

Proof. This is Theorem 4.6 in [McM]. □

CHAPTER 2

The Hausdorff measure

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2.1 The Hausdorff dimension and measure

Let (X, μ) be a metric space. The fundamental idea behind the definition of the Hausdorff dimension is that in a space of dimension d the "size" of a subset should increase like r^d, where r is a linear dimension of the set (the diameter, for example).

Definition 2.1. Given a set $Y \subseteq X$, its *diameter* is:

$$\operatorname{diam} Y := \sup_{x,y \in Y} \mu(x, y).$$

Definition 2.2. For $\delta > 0$ we say that a countable collection of subsets $\{U_i\}_{i\in\mathbb{N}}$ of X is a *δ-cover* of X if $X \subseteq \bigcup_i U_i$ and for each i we have that $\operatorname{diam} U_i \leqslant \delta$.

Now, for $\delta > 0$ and $d > 0$ define

$$\mathcal{H}^d_\delta(X) := \inf \left\{ \sum_{i\in\mathbb{N}} (\operatorname{diam} U_i)^d \;\middle|\; \{U_i\}_i \text{ is a } \delta\text{-cover of } X \right\}. \tag{2.1}$$

As δ decreases to zero, the number of δ-covers decreases, so the infimum increases. It thus makes sense to define the d-dimensional Hausdorff measure as

$$\mathcal{H}^d(X) := \sup_{\delta\to 0} \mathcal{H}^d_\delta(X).$$

Intuitively, the operation of taking the infimum in (2.1) corresponds to looking for the covering that best fits the set X without too many superpositions among the U_i while keeping the diameters small.

Of course the d-dimensional measure can be 0 or ∞. Indeed, this will be quite a frequent case: we will have to choose d carefully to have some meaningful measure, because if d is too high the measure will be zero and, on the other hand, if it is too low the measure will be infinite. From the definition it follows quite easily that the function $d \mapsto \mathcal{H}^d(X)$ is decreasing. Actually there is a stronger result: suppose that for some d we have $\mathcal{H}^d(X) < \infty$ and let $d' > d$; then for some small δ there is a δ-cover $\bigcup_i U_i$ of X such that $\sum_i (\operatorname{diam} U_i)^d < \mathcal{H}^d(X) + \varepsilon < \infty$ (for some $\varepsilon > 0$). Then:

$$\sum_i (\operatorname{diam} U_i)^{d'} = \sum_i (\operatorname{diam} U_i)^d (\operatorname{diam} U_i)^{d'-d} \leqslant \delta^{d'-d} \sum_i (\operatorname{diam} U_i)^d .$$

But we can choose δ arbitrarily small: we then conclude that $\mathcal{H}^{d'}(X) = 0$.

It follows then that the function $d \mapsto \mathcal{H}^d(X)$ (except when it is identically zero or infinite) has exactly one discontinuity h. Below it it is infinite, above it is zero. At the point of the discontinuity there is an “equilibrium”.

Definition 2.3. The *Hausdorff dimension* $\operatorname{H-dim} X$ of a metric space X is the number h found as above. In the special cases when the function $d \mapsto \mathcal{H}^d(X)$ is identically zero or infinite, we respectively set $\operatorname{H-dim} X = \infty$ or $\operatorname{H-dim} X = 0$. Equivalently we can define it as the number $\inf \{ d \geqslant 0 \mid \mathcal{H}^d(X) = 0 \}$, with the convention that the infimum of an empty set is $+\infty$.

From the definition it is rather easy to prove the following lemmas.

Lemma 2.4. *The Hausdorff dimension is monotone; that is, if $Y \subseteq X$ (with the induced metric structure), then*

$$\operatorname{H-dim} Y \leqslant \operatorname{H-dim} X. \tag{2.2}$$

This immediately implies that, for a family of sets $\{X_i\}_{i\in I}$,

$$\operatorname{H-dim} \bigcup_{i\in I} X_i \geqslant \sup_{i\in I} \operatorname{H-dim} X_i, \tag{2.3}$$

with the following remarks:

- *let $\{X_i\}$ be finite or countable family of Borel sets for X: then the equality holds;*
- *otherwise, it is possible to achieve the strict inequality.*

Proof. If $Y \subseteq X$, every δ-cover of X is also a δ-cover of Y, so, by definition, for each δ and d we have that $\mathcal{H}^d_\delta(Y) \leqslant \mathcal{H}^d_\delta(X)$. This implies that $\mathcal{H}^d(Y) \leqslant \mathcal{H}^d(X)$ for each d, so the unique discontinuity point that defines the Hausdorff dimension is smaller for Y than for X, which is equation (2.2). Equation (2.3) immediately follows.

The second part descends from the well-known fact that the d-dimensional Hausdorff measure $\mathcal{H}^d$ is σ-subadditive on Borel sets (i.e., it is actually a measure on at least the Borel sets; see, for example, [Rog], particularly Theorem 27); let $d = \sup_i \operatorname{H-dim} X_i$: then for each $d' > d$ and $i \in I$ we have $\mathcal{H}^{d'}(X_i) = 0$ and, by σ-subadditivity, $\mathcal{H}^{d'}(\bigcup_i X_i) = 0$, that means $\text{H-dim} \bigcup_i X_i \leqslant d'$.

For the last assertion it is enough to notice that a point has zero dimension, so each finite or countable set has zero dimension. Thus any decomposition of a positive dimension set as union of at most countable sets is a valid counterexample. □

Lemma 2.5. *Let X and Y be two metric spaces and $f\colon X \to Y$ a surjective, α-Hölder continuous function. Then*

$$\text{H-dim}\, Y \leqslant \frac{1}{\alpha}\, \text{H-dim}\, X.$$

In particular, if f is a locally bi-Lipschitz homeomorphism we have exactly

$$\text{H-dim}\, X = \text{H-dim}\, Y.$$

Proof. By hypothesis there are $\alpha \in (0, 1]$ and $C > 0$ such that

$$|f(x) - f(y)| \leqslant C \cdot |x - y|^\alpha \qquad x, y \in X.$$

Let $\{U_i\}$ be a δ-cover for X. Let $\delta' := C \cdot \delta^\alpha$ and, for $d \geqslant 0$, let $d' := d/\alpha$ (note that $\delta' \to 0$ as $\delta \to 0$). Then, taking $Y_i := f(X_i)$, because of the surjectivity of f we have that $\{Y_i\}$ is a δ'-cover of Y. Then

$$\mathcal{H}^{d'}_{\delta'}(Y) \leqslant \sum_{i\in I} (\text{diam}\, f(U_i))^{d'} \leqslant \sum_{i\in I} (C \cdot (\text{diam}\, U_i)^\alpha)^{d/\alpha} = C^{d/\alpha} \sum_{i\in I} (\text{diam}\, U_i)^d,$$

where the expression on the right-hand side is one of the elements that appear in the infimum that defines $\mathcal{H}^d_\delta(X)$. Taking δ-covers $\{U_i\}$ such that the right-hand side approximate arbitrarily well the number $\mathcal{H}^d_\delta(X)$ we have that $\mathcal{H}^{d'}_{\delta'}(X) \leqslant C^{d/\alpha}\mathcal{H}^d_\delta(Y)$. Passing to the limit for $\delta \to 0$ it follows that $\mathcal{H}^{d'}(Y) \leqslant C^{d/\alpha}\mathcal{H}^d(X)$.

Now specialize d to some number greater than $\text{H-dim}\, X$: then $\mathcal{H}^{d'}(Y) \leqslant \mathcal{H}^d(X) = 0$, so $\text{H-dim}\, Y \leqslant d' = d/\alpha$. Letting $d \searrow \text{H-dim}\, X$ we obtain $\text{H-dim}\, Y \leqslant \frac{1}{\alpha}\, \text{H-dim}\, X$.

The second equation follows immediately (a Lipschitz continuous function is by definition a 1-Hölder continuous function). □

Remark 2.6. When $d \in \mathbb{N}$, the d-dimensional Hausdorff measure on $\mathbb{R}^d$ coincides (up to a multiplicative factor) with the d-dimensional external Lebesgue measure.

Remark 2.7. At the point of discontinuity h, the Hausdorff measure can assume any value, including zero and infinity. Should the latter cases happen, one could generalize even more this construction, changing the scale function r^d (also called the *gauge function*) with other finer functions (such as $r^d \log r$ or similar), trying to find the exact point where we have a positive and finite measure. However, this procedure is completely out of the scope of this work. In [Rog] many such generalizations are discussed.

2.2 Hyperbolic sets and cookie cutter maps

It would be difficult to compute the Hausdorff dimension of the Mandelbrot set directly using the given definition. Instead, we will make use of some intermediate results. As we will see in section 2.3, the dimension of the Mandelbrot set can be determined by seeking adequately large Julia sets; in turn, we can estimate the dimension of a Julia set employing hyperbolic sets and cookie cutter maps that are defined here.

Definition 2.8 (hyperbolic set)**.** Let f be a rational map of the sphere and let $X \subset \hat{\mathbb{C}}$ be a closed and forward-invariant set for f. We say that X is an *hyperbolic set* if there are numbers $C > 0$ and $k > 1$ that verify the following condition:

$$\left|(f^{\circ n})'(z)\right| \geqslant C \cdot k^n \qquad \forall n \in \mathbb{N}, z \in X$$

(with respect to the spherical metric).

We will also say that the map f is *expanding* on X.

From the definition it is quite clear that a hyperbolic set is contained in the Julia set for f, as any subsequence of $\{f^{\circ n}\}_n$ converging on some open set U that intersects X should have converging derivatives. Thus it will be helpful, in order to estimate the dimension of the Julia set, to give the following definition.

Definition 2.9. Let f be a holomorphic dynamical system: then we define its *hyperbolic dimension* as

$$\text{hyp-dim}\, f := \sup_Y \text{H-dim}\, Y,$$

where Y ranges over all the hyperbolic sets for f.

The discussion above immediately implies that $\text{H-dim}\, J(f) \geqslant \text{hyp-dim}\, f$. We are now interested in some method for building sufficiently large hyperbolic sets.

Definition 2.10 (cookie cutter map)**.** Let $U \subset \hat{\mathbb{C}}$ be a simply connected, open subset of the sphere that omits at least three points and let $U_1, \ldots, U_N \subset U$ be pairwise disjoint, simply connected and open sets with closure in U. We say that the holomophic dynamical system $f \colon U \to U$ is a *cookie cutter map* (or even only *cookie cutter*) if for each $i = 1, \ldots, N$ there is a natural number n_i such that $f^{\circ n_i}$ is a bijective from U_i onto U.

Using cookie cutters we can build hyperbolic sets in a way that is a generalization of the procedure used to construct the Cantor set. Let $\tau_i := (f^{\circ n_i}|_{U_i})^{-1} \colon U \to U_i$. Being an holomorphic map between two simply connected hyperbolic surfaces, one of which has closure strictly contained in the other, it must be a contraction (more specifically, for each compact set $K \subset U$ there is some constant $c_K < 1$ such that for $x, y \in K$ it results $d(f(x), f(y)) \leqslant c_K d(x, y)$, where d is the Poincaré metric for the two surfaces; the proof of this fact is in [Mil], Theorem 2.11).

Now, for each subset $V \subseteq U$ we can consider

$$\alpha(V) := \tau_1(V) \cup \cdots \cup \tau_N(V) \subset U.$$

Let also

$$X_0 := \bigcap_{k \in \mathbb{N}} \alpha^{\circ k}(\bar{U}_1 \cup \cdots \cup \bar{U}_N),$$

$$X := \bigcup_{k \in \mathbb{N}} f^{\circ k}(X_0) = X_0 \cup f(X_0) \cup \cdots \cup f^{\circ M-1}(X_0) \qquad (M := \max_i n_i).$$

The set X is called *repeller* for the function f. It is rather interesting, because we can give a good estimate to its Hausdorff dimension.

Lemma 2.11. *The set X is hyperbolic for f.*

Proof. X is evidently closed and f-forward invariant. We just have to show that f is expanding on it.

It is clear that, for each i, $f^{\circ n_i}|_{U_i} = \tau_i^{-1}$ is expanding on the compact set $X_0 \cap U_i$, since τ_i is contracting (note that the Poincaré metric of U is equivalent to the spherical metric on X_0).

The general case can be established factorizing $f^{\circ n}(z)$ as follows (for $z \in X$): first make $j_1 < M$ iterations so that $f^{\circ j_1}(z) \in X_0 \cap U_{i_1}$ for some i_1; then do n_{i_1} iterations, to land on $(f^{\circ n_{i_1}}|_{U_{i_1}}) \circ f^{\circ j_1} \in X_0 \cap U_{i_2}$, for some i_2. Continue on by doing each time n_{i_h} iterations of f until $j_1 + n_{i_1} + \cdots + n_{i_k} > n - M$ and set

$$j_2 := n - (j_1 + n_{i_1} + \cdots + n_{i_k}).$$

We then have that, in some neighbourhood of z,

$$f^{\circ n} = f^{\circ j_2} \circ (f^{\circ n_{i_k}}|_{U_{i_k}}) \circ \cdots \circ (f^{\circ n_{i_1}}|_{U_{i_1}}) \circ f^{\circ j_1};$$

they're all expanding, except maybe $f^{\circ j_2}$ and $f^{\circ j_1}$, that account for at most $2M$ iterations, independently of n. Thus, f is expanding. □

Lemma 2.12. *Let $\delta :=$ H-dim X_0. Then*

$$\text{hyp-dim}\, f \geqslant \text{H-dim}\, X \geqslant \delta \geqslant \frac{\log N}{\log(\max_i \sup_{U_i} \left|(f^{\circ n_i})'\right|)}.$$

Proof. The first two inequalities are obvious.

For the last one consider the dynamical system g defined on X_0 defined by the formula

$$g|_{X_0 \cap U_i} = f^{\circ n_i}|_{X_0 \cap U_i} \qquad i = 1, \ldots, N.$$

Evidently $g(X_0) \subseteq X_0$ and g has X_0 as repeller. We can thus apply the well-known Bowen's formula (see [Bow] and [BKZ]), which asserts that δ is the unique non negative real number that satisfies:

$$0 = \lim_{k \to \infty} \frac{1}{k} \log \left(\sum_{w \in g^{\circ(-k)} \cap X_0} \left|(g^{\circ k})'(w)\right|^{-\delta} \right),$$

for any $w \in X_0$. Exponentiating, we have:

$$\begin{aligned} 1 &\geqslant \lim_{k \to \infty} \left(N^k \sup_{w \in \bigcup_i U_i} \left|(g^{\circ k})'(w)\right|^{-\delta} \right)^{\frac{1}{k}} \\ &= N \cdot \lim_{k \to \infty} \left(\sup_{w \in \bigcup_i U_i} \left|(f^{\circ n_{j_k}} \circ \cdots \circ f^{\circ n_{j_1}})'(w)\right|^{-\delta} \right)^{\frac{1}{k}} \\ &\geqslant N \cdot \lim_{k \to \infty} \left(\max_i \sup_{w \in U_i} \left|(f^{\circ n_i})'(w)\right|^{-k\delta} \right)^{\frac{1}{k}} \\ &= N \cdot \max_i \sup_{w \in U_i} \left|(f^{\circ n_i})'(w)\right|^{-\delta}, \end{aligned}$$

where $j_1, \ldots, j_k$ depend on w and are taken tracing the orbit of w so that the previous expressions are meaningful.

Taking again the logarithms the thesis immediately follows. □

2.3 Relationship between Mandelbrot and Julia sets

Let $\lambda \in \partial M$: we now want to show a relation between the dimension of the Julia set $J(P_\lambda)$ and that of the Mandelbrot set itself. The result we will work out is highly non-trivial, since the two sets are obtained with completely different methods (the first naturally "lives" in the dynamical system's space, while the second is a parameter set), so there is no *a priori* link between them. Surprisingly it turns out that many properties of Julia sets are transferable to the Mandelbrot set.

Theorem 2.13. *Let $\{f_\lambda\}_\lambda$ be a family of rational maps analytically depending on λ ranging on an open subset $\Lambda \subseteq \mathbb{C}$. Let f_{λ_0} be a J-unstable element in this family. Then we have:*

$$\text{H-dim}\,\{\, \lambda \in \Lambda \mid f_\lambda \textit{ is } J\textit{-unstable} \,\} \geqslant \text{hyp-dim}\, f_{\lambda_0}.$$

To prove it, let's start with a definition.

Definition 2.14 (holomorphic motion)**.** Let $\Lambda \subseteq \mathbb{C}$ be an open set with a base point $\lambda_0 \in \Lambda$ and $X \subseteq \mathbb{C}$ open too. An *holomorphic motion* is a family of maps $i_\lambda \colon X \to \mathbb{C}$ $(\lambda \in \Lambda)$ that verifies the following properties:

- i_λ is a homeomorphism with its image for each λ;
- $\lambda \mapsto i_\lambda(z)$ is analytic for each $z \in X$;
- i_{λ_0} is the identical function of X.

We will also use the notation $X_\lambda = i_\lambda(X)$.

The goal of this definition is to express the concept of a set changing with respect to some (complex) time, but essentially preserving its structure. We immediately apply this interpretation to hyperbolic sets. The following lemma is well-known; in [Aba] there is a proof for the same result in the case of real dynamical systems: the proof in our case is completely analogous.

Lemma 2.15. *Let f_0 be a rational function of the Riemann sphere and X a hyperbolic set for f_0. Then X is stable under perturbations of f_0, i.e., there is a holomorphic motion i_f of X defined on a neighbourhood of f_0 (in the space of rational function with the same degree) such that each function f near to f_0 has a hyperbolic set X_f and i_f conjugates (X, f_0) with (X_f, f) (in formulae: $X_f = i_f(X)$ and $i_f \circ f_0 = f \circ i_f$).*

We're now ready to state and prove a few other lemmas, then prove Theorem 2.13.

Lemma 2.16. *A holomorphic motion i_λ defined on $|\lambda| < R$ is bi-Hölder continuous with exponent $\alpha(|\lambda|/R)$, where α is a universal function and $\alpha(t) \nearrow 1$ as $t \searrow 0$. Moreover, the multiplicative constant of the Hölder relation is universally limited.*

Proof. As proved in [BR] and [ST], a holomorphic motion is a $K(|\lambda|/R)$-quasiconformal function, with $K(t) \searrow 1$ as $t \searrow 0$. At the same time it follows from Mori's formula (see, for instance, [Ahl]) that a K-quasiconformal function is $1/K$-Hölder continuous with uniform multiplicative constant (equal to 16). □

Putting together lemmas 2.5, 2.15 and 2.16 we have that for each k the functions set $\{ f \mid \text{hyp-dim} f > k \}$ is open (or, equivalently, $f \mapsto \text{hyp-dim} f$ is lower semi-continuous).

The following is the key lemma for proving 2.13. Let us indicate with $B_r(z)$ the (open) ball of radius r centered in z and with $D_r(z) = \bar{B}_r(z)$ the (closed) disc with same radius and center.

Lemma 2.17. *Let i_λ be a holomorphic motion on X with base point 0, where λ ranges over the open unit disc $\Delta := B_1(0)$. Let also $v\colon \Delta \to \hat{\mathbb{C}}$ be an holomorphic map such that $z_0 := v(0) \in X$, but $v(\lambda) \neq i_\lambda(z_0)$ for $\lambda \neq 0$. Then:*

$$\text{H-dim}\,\{ \lambda \in \Delta \mid v(\lambda) \in X_\lambda \} \geqslant \lim_{r\to 0} \text{H-dim}(X \cap D_r(z_0)). \tag{2.4}$$

Proof. Up to a Möbius change of coordinate depending analytically on λ, we can assume that $z_0 = 0$ and $i_\lambda(0) \equiv 0$. First consider the case when $v'(0) \neq 0$: then there are $a, \rho > 0$ such that v is injective in $B_\rho(0)$ with $a|\lambda| < |v(\lambda)| < \infty$ for $\lambda \in B_\rho(0)$. Take then

$$b_r := \sup \{ |i_\lambda(z)| \mid z \in X \cap D_r(0), \lambda \in D_\rho(0) \}.$$

Since by Lemma 2.16 the maps i_λ are equicontinuous, $b_r \to 0$ when $r \to 0$; so there is some positive r_0 for which $b_r < a\rho$ when $0 < r < r_0$.

For such an r, $z \in X \cap D_r(0)$ and $|\mu| < R_r := a\rho/b_r$, consider the equation

$$v(\lambda) - i_{\lambda\mu}(z) = 0 \qquad \text{where } \lambda \in \Delta_\mu := B_{\min\{\rho,\rho/|\mu|\}}(0) \tag{2.5}$$

(note that the constructed R_r tends to ∞ for $r \to 0$).

By construction, on $\partial\Delta_\mu$ we have $|v(\lambda)| > |i_{\lambda\mu}(z)|$, so, by Rouché's theorem, for fixed z and μ, equation (2.5) has exactly one solution (becase $v(\lambda) = 0$ has exactly one solution by hypothesis), that depends analytically on μ (because of Dini's theorem). Moreover, for fixed μ, changing z leads to a different solution for λ, since i_λ is injective.

We now have to build a holomorphic motion that connects the two sides of equation (2.4). Define:

$$Y^r_\mu := \{ \lambda \in \Delta_\mu \mid v(\lambda) = i_{\lambda\mu}(z) \text{ for some } z \in X \cap D_r(0) \}.$$

Clearly

$$Y^r_0 = v^{-1}(X \cap D_r(0)), \qquad\qquad Y^r_1 \subseteq \{ \lambda \in \Delta \mid v_\lambda \in i_\lambda(X) \},$$

where v^{-1}, being analytical, does not change the Hausdorff dimension. From the discussion above we can also see that Y^r_μ is a holomorphic motion with respect to the variable μ when $|\mu| < R_r$, where the injection $j^r_\mu(v^{-1}(z))$ is given by the unique solution of equation (2.5).

Using again lemma 2.16 and 2.5 it turns out $\text{H-dim}\, Y^r_1 \geqslant \alpha(1/R_r) \cdot \text{H-dim}\, Y^r_0$, that easily implies the thesis for $r \to 0$.

We just left out the case $v'(0) = 0$, that can be reconducted to the previous one this way: let m be the order of zero of v in 0 and $G(z) := z^m$. Perhaps after a change of coordinate, we have that $\infty \in X$ and $i_\lambda(\infty) \equiv \infty$. Then let $\tilde{X} = G^{-1}(X)$ and $\tilde{X}_\lambda = G^{-1}(X_\lambda)$.

Since G is a covering branched only over 0 and ∞, we can lift v to $\tilde{v}\colon \Delta \to \hat{\mathbb{C}}$ and i_λ to $\tilde{i}_\lambda\colon \tilde{X} \to \tilde{X}_\lambda$. Now $\tilde{v}$ and $\tilde{i}_\lambda$ satisfy the hypoteses of the lemma and $\tilde{v}'(0) \neq 0$, so the first part of the proof applies.

Moreover $v = G \circ \tilde{v}$ and $i_\lambda \circ G = G \circ \tilde{i}_\lambda$, so, by lemma 2.5:

$$\left\{ \lambda \in \Delta \,\middle|\, \tilde{v}(\lambda) \in \tilde{X}_\lambda \right\} = \{ \lambda \in \Delta \mid v(\lambda) \in X_\lambda \}$$
$$\text{H-dim}(X \cap D_r(0)) = \text{H-dim}(\tilde{X} \cap G^{-1}(D_r(0))).$$

The proof is thus complete. □

Proof of theorem 2.13. Fix $\varepsilon > 0$ and consider some set X, hyperbolic for f_{λ_0} and with $\text{H-dim}\, X > \text{hyp-dim}\, f_{\lambda_0} - \varepsilon$. Since X is compact there is some point $z_0 \in X$ with

$$\lim_{r \to 0} \text{H-dim}(X \cap D_r(z_0)) = \text{H-dim}\, X$$

(by virtue of lemma 2.4 in the finite case).

Because of the stabiliy of hyperbolic sets, we can choose some neighbourhood $\Lambda' \subseteq \Lambda$ of λ_0 and a holomophic motion i_λ of X with $\lambda \in \Lambda'$, such that $X_\lambda = i_\lambda(X)$ is hyperbolic for f_λ and $i_\lambda \circ f_{\lambda_0} = f_\lambda \circ i_\lambda$. Perhaps after reducing Λ', we may also assume that for $\lambda \in \Lambda'$ we have that

$$\lim_{r \to 0} \text{H-dim}(X_\lambda \cap D_r(i_\lambda(z_0))) > \text{H-dim}\, X - \varepsilon$$

(since, as we already mentioned, the hyperbolic dimension is lower semi-continuous) and that the critical points of f_λ do not bifurcate in Λ', except maybe at λ_0.

Recall now Lemma III.2 in [MSS]: let Λ' be an open simply connected set of parameters and $\varphi\colon \Lambda' \to \hat{\mathbb{C}}$ an analytic function such that for $\lambda \in \Lambda'$ the point $\varphi(\lambda)$ doesn't belong to any f_λ-forward orbit of a critical point for f_λ; suppose also that either $\varphi(\lambda)$ is not f_λ-periodic for any $\lambda \in \Lambda'$ or that for some $N \geqslant 0$ we have $f_\lambda^{\circ N}(\varphi(\lambda)) = \varphi(\lambda)$ for all $\lambda \in \Lambda'$: then λ is J-stable for all $\lambda \in \Lambda'$.

Set $\varphi(\lambda) := i_\lambda(z_0)$: the thesis of such lemma is false by hypothesis. On the other hand it is trivially true that either $i_\lambda(z_0)$ is not f_λ-periodic for any $\lambda \in \Lambda'$ or there is some $N \geqslant 0$ such that $f_\lambda^{\circ N}(i_\lambda(z_0)) = i_\lambda(z_0)$ for each $\lambda \in \Lambda'$, since i_λ conjugates the dynamics of f_λ on X_λ (which is f_λ-forward invariant). We then have, by contradiction, that there must be some $\lambda_1 \in \Lambda' \setminus \{\lambda_0\}$ such that $f_{\lambda_1}^{\circ N}(c) = i_{\lambda_1}(z_0)$ for $N > 0$, where c is a critical point of f_{λ_1} (there is no loss of generality assuming that $\lambda_1 \neq \lambda_0$, since lemma III.2 can be applied to any J-unstable point of Λ').

So there is a whole branch of critical points c_λ of f_λ, where λ ranges in a neighbourhood $\Lambda'' \subseteq \Lambda'$ of λ_1 and $c_{\lambda_1} = c$. Moreover, since for any z we have $f_\lambda^{\circ N}(c_\lambda) \not\equiv i_\lambda(z)$, Λ'' can be chosen small enough so that $f_\lambda^{\circ N} \neq i_\lambda(z_0)$ for $\lambda \in \Lambda'' \setminus \{\lambda_1\}$.

Lemma 2.17 with $v(\lambda) := f_\lambda^{\circ N}(c_\lambda)$ (and after some appropriate affine coordinate change) implies then that:

$$\begin{aligned} \text{H-dim}\left\{ \lambda \in \Lambda'' \,\middle|\, f_\lambda^{\circ N}(c_\lambda) \in X_\lambda \right\} &\geqslant \lim_{r \to 0} \text{H-dim}(X_{\lambda_1} \cap D_r(i_{\lambda_1}(z_0))) \\ &> \text{hyp-dim}\, f_{\lambda_0} - 2\varepsilon. \end{aligned}$$

Recalling again that $f_\lambda^{\circ N}(c_\lambda) \not\equiv i_\lambda(z)$ for each z, one has that the condition on the left-hand side implies that λ in J-unstable. Since ε is arbitrary, the theorem is thus proved. □

CHAPTER 3

Main result

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3.1 Main result

We're now ready to state our main result.

Theorem 3.1 (Dimension of the Mandelbrot set's boundary). *Let M be the Mandelbrot set. Then:*

$$\text{H-dim}(\partial M) = 2.$$

Remark 3.2. Actually a much stronger result can be proven: each non-empty open set of ∂M has Hausdorff dimension 2. The proof of this assertion is completely similar to that of Theorem 3.1.

The great complexity expressed by having a boundary of dimension two is thus a property valid not only for the whole set, but even locally. In other words, that property is homogeneus in the set: there are no regions where we can observe a more regular structure.

The proof for this fact will be given in two different stages: first, we will prove a similar statement for Julia sets (we will find, for appropriate parameters $\lambda \in \partial M$, hyperbolic sets for P_λ of dimension arbitrarily close to 2); then, we will transplant the result from the dynamic space (where the Julia set is defined) to the parameter space (where the Mandelbrot set is defined), using the theory developed in chapter 2. Such two-stages approach for showing properties for the Mandelbrot set appears quite frequently, for instance in [Hub].

3.2 Fixed and parabolic points

Let f be a holomorphic dynamical system.

Definition 3.3 (fixed and parabolic points)**.** A point $z_0 \in \operatorname{dom} f$ is a *fixed point* if $f(z_0) = z_0$. If this happens, the f can be locally written as

$$f(z) = z_0 + a_1(z - z_0) + (z - z_0)^{n+1} \cdot (a_{n+1} + a_{n+2}(z - z_0) + \dots) \qquad a_n \neq 0 \tag{3.1}$$

for some number $n \geqslant 1$. Then we say that $a_1 = f'(z_0)$ is the *multiplier* of f in the point z_0; when $a_1 \neq 0$, we can write it as $a_1 = \exp(2\pi i\alpha)$, where α is called the *rotation number* and is defined up to addition of integer numbers (the rotation number is particularly meaningful when $|a_1| = 1$, but we will consider it defined also when this is not the case).

Furthermore, the fixed point z_0 is called *indifferent* if it has a real rotation number, *parabolic* if it has a rational rotation number and *tangent to the identity* if the rotation number is zero (that is, the multiplier is 1). In this last case it is meaningful the number $n + 1$: we will call it the *multiplicity* of the parabolic point.

Note that the definitions given above are completely local and thus are also valid for function germs.

Definition 3.4 (basins and immediate basins)**.** With f a holomorphic dynamical system and $\zeta \in \operatorname{dom} f$ a parabolic fixed point with rotation number p/q, we define the *parabolic basin* of ζ as the set

$$\mathcal{B} = \left\{ z \in \operatorname{dom} f \;\middle|\; \begin{array}{l} \text{there is a neighbourhood of } z \text{ on which, for each } n,\ f^{\circ n} \text{ is} \\ \text{defined and converges uniformly to } \zeta \text{ as } n \to \infty. \end{array} \right\}.$$

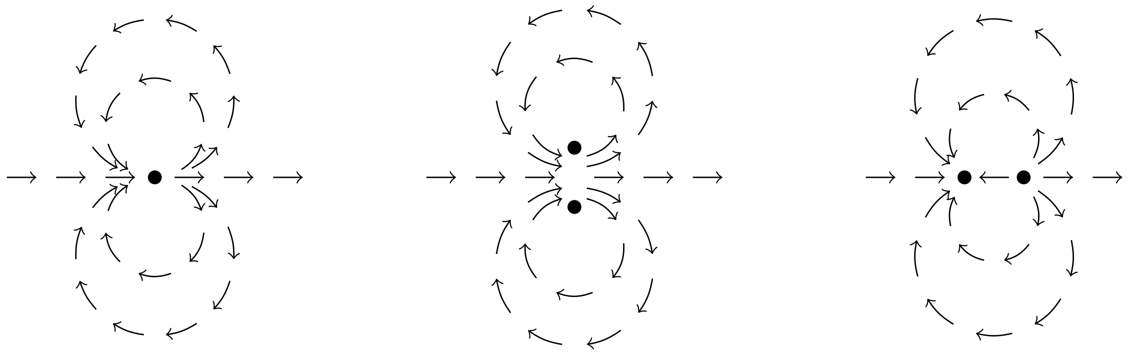
A connected component $\mathcal{B}' \subset \mathcal{B}$ is said to be an *immediate parabolic basin* if it is periodic (there is k such that $f^{\circ k}(\mathcal{B}') = \mathcal{B}'$) and $f^{\circ k}|_{\mathcal{B}'}$ is proper (thus it is a branched covering). In general there can be more than one immediate parabolic basin: when we'll refer to *the* immediate parabolic basin we will mean the union of all the immediate parabolic basins.

3.3 Perturbation of a parabolic point

In the continuation of this work we will be particularly concerned with the behaviour of a dynamical system in the neighbouhood of a parabolic point and, particularly, with what happens when we change a bit the function that defines the dynamical system.

Let's assume that f_0 is tangent to the identity in the fixed point 0 with multiplicity $n + 1$ (the case of 0 being parabolic can be brought back to the case tangent to identity considering a suitable iteration of f_0): then 0 is a root of order $n + 1$ of the function $f_0(z) - z$. If we perturb slightly f_0 to f, generally the multiple root will be exploded in $n + 1$ simple roots, that will be simple fixed points for the dynamical system f. In the particular case $n + 1 = 2$, one can see that the behaviours of orbits near the parabolic point (depicted in figure 3.1a) depends on whether the splitting of the parabolic point leads to two indifferent points (figure 3.1b) or one attractive point and one repulsive point (figure 3.1c).

In the end we'll be able to prove the following theorem (the most important in this work).



(a) Original

(b) Two indifferent points

(c) One attracting and one repelling point

Figure 3.1: Phase portraits of perturbed and non-perturbed parabolic points

Theorem 3.5 (dimension of Julia sets)**.** *Let f_0 be a rational map of degree $d \geqslant 2$ and ζ a parabolic fixed point for f_0 with rotation number p/q. Suppose that the immediate parabolic basic of ζ contains exactly one critical point of f_0.*

Then, for $\varepsilon > 0$ and $b \geqslant 0$ there is a neighourhood $\mathcal{N}$ of f_0 (in the space of rational maps of degree d), a neighbourhood $V \ni \zeta$ and natural numbers N_1 and N_2 that verify the following condition: let $f \in \mathcal{N}$ with a fixed point in V and rotation number α that, for some integers $a_1 \geqslant N_1$ and $a_2 \geqslant N_2$ and some $\beta \in [0,1) + i[-b,b]$, satisfies

$$q\alpha = p \pm \frac{1}{a_1 \pm \frac{1}{a_2 \pm \beta}};$$

then

$$\text{hyp-dim}\, f > 2 - \varepsilon.$$

While the thesis of this theorem is quite clear, its hypotheses (particularly the allowed values for α) appear technical and obscure. A better explanation of its meaning will be possible after having shown a few definitions and constructions that can be made around a parabolic point (see chapter 4). The proof of theorem 3.5 will occupy most of the following chapters.

Remark 3.6. It is known that every parabolic point of a rational map always has at least one critical point in its immediate parabolic basin. Thus, to apply theorem 3.5 it is enough to verify that there is at most one.

3.4 Proof of the main result

Using theorems 3.5 and 2.13 we can show the main theorem, 3.1.

Proof of the main result. The polynomials of type $P_\lambda(z) = z^2 + \lambda$ have exactly one critical point, so satisfy the hypotheses of theorem 3.5. Pick then $\lambda = \frac{1}{4}$: it is easy to see that P_λ has one parabolic fixed point (which is $\frac{1}{2}$ and is tangent to the identity, with multiplicity

$n + 1 = 2$). Since it has an indifferent periodic point, it is J-unstable (by Theorem 4.8 in [McM]).

By perturbing slightly P_λ to f, we can make it have a fixed point with rotation number α as requested by the same theorem, since $p/q = 1$ is an accumulation point for the allowed values for α. Furthermore, we can, without loss of generality, ask that $\Im\beta = 0$, thus making $\alpha \in \mathbb{R}$ and the new fixed point indifferent. As before, this means that f is J-unstable itself.

Using theorem 2.13 and proposition 1.18 we have, at last, that H-dim $\partial M > 2 - \varepsilon$ for each $\varepsilon > 0$, which is our thesis. □

Remark 3.7. The proof of the stronger result stated in Remark 3.2 follows the same structure of the above, using the fact that for λ in a dense set of ∂M it happens that P_λ has a parabolic periodic point.

CHAPTER 4

Parabolic points

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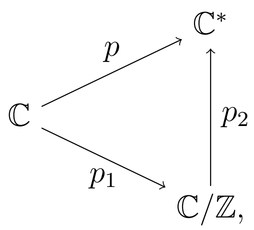
4.1 Some notation conventions

Before starting the chapter, we agree to call τ_0 the function $w \mapsto -w^{-1}$ and, for some function f that takes values in $\hat{\mathbb{C}}$, $f^* := \tau_0 \circ f$ (note that $\tau_0 = \tau_0^{-1}$).

Consider now the following Riemann surfaces: the plane $\mathbb{C}$, the cylinder $\mathbb{C}/\mathbb{Z}$ and the cylinder $\mathbb{C}^*$. Between them the following maps are defined:



where

$$p(z) := \exp(2\pi i z),$$

p_1 is the quotient projection and p_2 makes the diagram commuting (of course, p_2 is a conformal isomophism between $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}^*$).

Finally, for $\eta \in \mathbb{R}$ we define

$$D(\eta) := \left\{ w \in \mathbb{C} \;\middle|\; |w - i\eta| \leqslant \frac{|\eta|}{4} \right\}$$
$$D'(\eta) := \left\{ w \in \mathbb{C} \;\middle|\; |w - i\eta| \leqslant \frac{|\eta|}{2} \right\}.$$

We will stick to these conventions for all the rest of this work.

4.2 The Leau-Fatou flower theorem

We want now to continue the discussion started with Section 3.2, investigating more deeply the local properties of a dynamical system near a point tangent to the identity. Consider the set

$$\mathcal{F} := \left\{ f\colon \operatorname{dom} f \to \hat{\mathbb{C}} \;\middle|\; f \text{ is analytic}, 0 \in \operatorname{dom} f \subseteq \hat{\mathbb{C}}, f(0) = 0 \right\} = \operatorname{End}(\mathbb{C}, 0);$$

let also $f_0 \in \mathcal{F}$ be a function tangent to the identity in the origin, with multiplicity $n+1$. Its Taylor series (3.1) can be rewritten as

$$f_0(z) = z + az^{n+1} + \dots.$$

There are a few known facts about how the dynamical system performs in its neighbourhood. For example, this point exhibits both attractive and repulsive behaviours, in different directions.

Let ξ be an n-th root of $\frac{1}{na}$ and let

$$v_j = \exp\left(2\pi i \frac{j}{2n}\right) \xi \qquad (0 \leqslant j < 2n),$$

so that the v_j are $2n$ evenly spaced vectors such that $nav_j^n = (-1)^j$. We call v_j a *repelling direction* (resp. *attracting direction*) when j is even (resp. odd).

The following theorem gives reasons for this nomenclature. The proof for this and for part of the following theorems is on [Mil], §10.

Proposition 4.1 (attracting and repelling directions)**.** *Let $z_0 \mapsto z_1 \mapsto \dots$ be an orbit for f_0, converging to zero non-trivially (i.e., without being definitively zero). Then the limit* $\lim_{k\to\infty} \sqrt[n]{k} z_k$ *exists and is exactly one of the attracting directions.*

On the contrary, let $z_0 \leftarrow\!\shortmid z_1 \leftarrow\!\shortmid \dots$ be a non trivial inverse orbit for f_0. Then the limit $\lim_{k\to\infty} \sqrt[n]{k} z_k$ *exists and is exactly one of the repelling directions.*

In other words, every orbit converging to a parabolic point reaches it tangentially to exactly one attracting direction (unless it's trivial), and vice versa for inverse orbits. This proposition can be furtherly refined after giving an important definition.

Definition 4.2 (petal)**.** Given a map f_0 tangent to the identity in the origin and an attractive direction v, the open set $\Omega \subseteq \hat{\mathbb{C}}$ is an *attracting petal* for the direction v if these conditions hold:

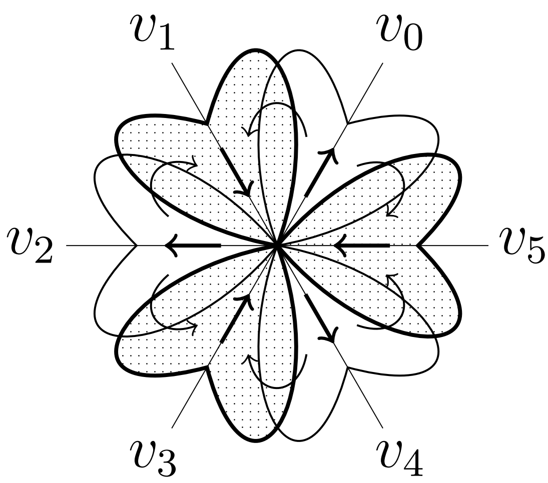


Figure 4.1: The Leau-Fatou flower

1. Ω is f_0-forward invariant;
2. an orbit of f_0 converges to the origin tangentially to v if and only if it belongs eventually to Ω.

A *repelling petal* is an attractive petal for f_0^{-1}.

Proposition 4.3 (Leau-Fatou flower). *Let f_0 be tangent to the identity in the origin, with multiplicity $n+1$; then within each neighbourhood of 0 there are $2n$ petals, one for each attracting and repelling direction (compare with figure 4.1, where the attracting petals have been drawn thicker and with a dotted background pattern).*

It is also possible to choose them with the following properties:

- *all the petals are simply connected;*
- *the union of all the petals is a punctured neighbourhood of the origin;*
- *for $n > 1$, each petal intersects only two other petals; precisely, the petal associated to the direction v_j ($j = 0, \ldots, 2n-1$) intersects the petals associated to the directions v_{j-1} and v_{j+1} and only them, where the sums are to be considered modulo $2n$; note that, of course, each petal only intersects petals with different attracting or repelling behaviour;*
- *in the special case $n = 1$, the situation degenerates a bit: there are only two petals, one repelling and one attracting, whose intersection has exactly two simply connected components;*
- *for each attracting petal Ω_+, $f^{\circ n}(\Omega_+) \to 0$ uniformly; similarly, for each repelling petal Ω_-, $f^{\circ(-n)}(\Omega_-) \cap \Omega_- \to 0$ uniformly.*

It easily follows that each attracting petal stays into the parabolic basin of the origin.

Corollary 4.4. *A parabolic point belongs to the Julia set. However, a parabolic point doesn't belong to its parabolic basin.*

Proof. It follows easily from the observation that each parabolic point has arbitrarily close points with different attracting or repelling behaviour. □

4.3 Fatou coordinates and Écalle cylinders

The petals described above are interesting not only because they give a rather good intuitive idea of the dynamics that arise near a parabolic point, but also because that the dynamic of f_0 on a petal can be *linearized*, i.e., it is conjugated to the shift map T of $\mathbb{C}$ defined by $T(w) := w + 1$.

We shall restrict to the case of a point tangent to the identity and with multiplicity $n + 1 = 2$. All the following results are perfectly valid for the general case of an arbitrary parabolic point, but require quite more complex notations. With this assumption we have only one attracting and one repelling petal, that will be indicated respectively with Ω_+ and Ω_-. Since we asked that the second derivative is not zero, we can, after a change of coordinate, assume it is 1.

Proposition 4.5 (Fatou coordinate for the repelling petal)**.** *Let $f_0 \in \mathcal{F}$, with $f_0'(0) = 1$ and $f_0''(0) = 1$. Then there exists an analytic function φ_0 that takes values in $\hat{\mathbb{C}}$, with the following properties:*

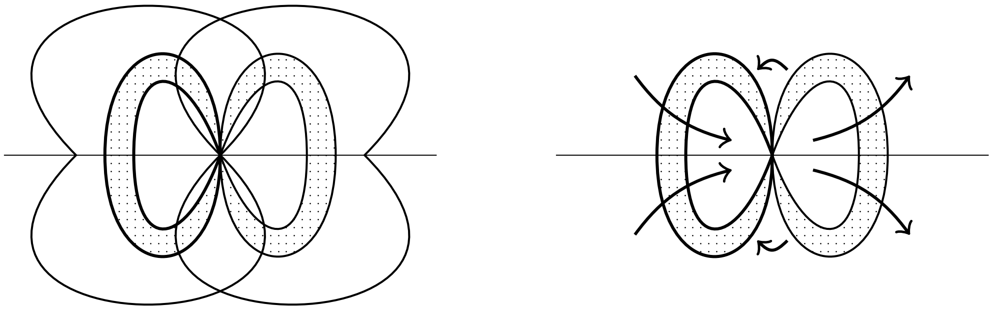
- $\operatorname{dom} \varphi_0 = \mathcal{Q}_0 \cup \{\, w \in \mathbb{C} \mid |\Re w| > \eta_0 \,\}$, *where*
$$\mathcal{Q}_0 = \left\{ w \in \mathbb{C} \;\middle|\; \arg(w + \xi_0) \in \left(\frac{\pi}{3}, \frac{5\pi}{3} \right) \right\},$$
for sufficiently large $\eta_0, \xi_0 > 0$;
- *when both sides are defined, $\varphi_0 \circ T = f_0 \circ \varphi_0$;*
- $\varphi_0(\mathcal{Q}_0) = \Omega_-$ *and φ_0 is injective on $\mathcal{Q}_0$;*
- $\varphi_0^*(w) = w + a \log w + b + o(1)$ *as $\mathcal{Q}_0 \ni w \to \infty$, for a and b constants.*

Proposition 4.6 (Fatou coordinate for the attracting petal)**.** *Let $f_0 \in \mathcal{F}$, with $f_0'(0) = 1$ and $f_0''(0) = 1$. Then there exist an analytic function Φ_0, defined over the parabolic basin $\mathcal{B}$ and taking values in $\mathbb{C}$, injective on $\Omega_+ \subset \mathcal{B}$ and such that $\Phi_0 \circ f_0 = T \circ \Phi_0$.*

The proofs for these two propositions can be found in [Mil], §10.

Remark 4.7. Note that the two coordinates are a bit asymmetrical: while the coordinate for the repelling petal is defined over the linearizing space and takes values into the dynamical system's state space, the one for the attracting petal is defined over the state space and takes values into the linearizing space.

This choice has made purely to ease the notation in the following discussion.



(a) Without suggested dynamics

(b) With suggested dynamics (petals removed for clarity)

Figure 4.2: The Écalle cylinders

One simple but really powerful consequence of having the Fatou linearizing coordinates is explained here: suppose to define over the attracting petal the relation $\sim$ such that

$$a \sim b \qquad \text{iff} \qquad \text{there is } k \geqslant 0 \text{ such that } f_0^{\circ k}(a) = b \text{ or } f_0^{\circ k}(b) = a$$

(i.e., $a \sim b$ when they belong to the same orbit); the quotient space $\Omega_+/\sim$ is conformally isomorphic to $\mathbb{C}/\mathbb{Z}$ via $(\Phi_0|_{\Omega_+})^{-1}$. The so-built surface is called the *Écalle cylinder* for the attracting petal. Similarly the Écalle cylinder for the repelling petal can be costructed. Moreover, the Écalle cylinder does not depend on the specific choice of petal for a certain attracting or repelling direction, since the intersection of two petals is still a petal and the inclusions of the two petals in their intersection induces on the quotient space an isomophism between the two cylinders.

4.4 The Écalle transformation

As suggested in figure 4.2, one can finds for each petal a *fundamental domain* that intersects each orbit converging into (or from) the petal in exactly one point. Gluing together the two sides of the fundamental domain the Écalle cylinder is obtained.

Figure 4.2b clearly suggests that the orbits of f induce a mapping from the "ends" of the repelling cylinder to the "ends" of the attracting cylinder. Exploiting such suggestion is not difficult once we have the Fatou coordinates.

Call $\tilde{\mathcal{B}} = \varphi_0^{-1}(\mathcal{B})$: obviously $T(\tilde{\mathcal{B}}) = \tilde{\mathcal{B}}$ (since $\mathcal{B}$ is forward f-invariant) and $\tilde{\mathcal{B}}$ contains $\{\, w \in \mathbb{C} \mid |\Im w| > \eta_0 \,\}$ for sufficiently large $\eta_0 > 0$ (since $\varphi_0(w)$ tends to zero when $w \to \infty$ while staying in $\mathcal{Q}_0$).

Then define $\tilde{\mathcal{E}}_{f_0} \colon \tilde{\mathcal{B}} \to \mathbb{C}$ such that $\tilde{\mathcal{E}}_{f_0} = \Phi_0 \circ \varphi_0$. The functional relations for φ_0 and Φ_0 immediately imply that $\mathcal{E}_{f_0}$ commutes with the map T: it is then well-defined the map

$$\mathcal{E}_{f_0} = p \circ \tilde{\mathcal{E}}_{f_0} \circ p^{-1} \colon p(\tilde{\mathcal{B}}) \to \mathbb{C}^*.$$

This map is called the *Écalle transformation.*

Proposition 4.8. *$\mathcal{E}_{f_0}$ extends analitically in 0 and ∞, taking respectively values 0 and ∞. Moreover, in both these points $\mathcal{E}_{f_0}$ has non-zero derivative.*

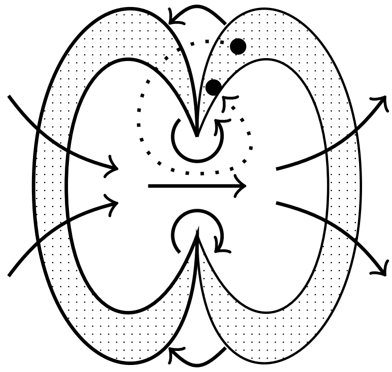


Figure 4.3: Phase portrait and Écalle cylinders for the perturbed parabolic point

Since we can freely replace Φ_0 with $\Phi_0 + c$ for some constant $c \in \mathbb{C}$ (while at the same time retaining all the properties of Φ_0), we can adjust c in order to make $\mathcal{E}'_{f_0}(0) = 1$.

4.5 Perturbation of the Écalle cylinders

Let us now consider what happens to the constructions outlined above when f_0 gets perturbated. We will consider perturbations contained in the set

$$\mathcal{F}_1 = \{ f \in \mathcal{F} \mid \alpha(f) \neq 0, |\arg \alpha(f)| < \pi/4 \} .$$

Pick $f \in \mathcal{F}_1$ near f_0: the double fixed point of f_0 has been splitted into two fixed points, one of which is 0; the other will be named $\sigma(f)$ and it tends to 0 as $f \to f_0$:

$$\sigma(f) = -f'(0)(1 + o(1)) = -2\pi i\alpha(f)(1 + o(1)) \qquad \text{for } f \to f_0.$$

At the same time, the structure of the Écalle cylinders changes a bit, as shown in Figure 4.3: the orbits that before tended to the parabolic point, now are allowed to "pass the gate" between the two new fixed points. The horn map now can be continued and become a *return map*, that it is actually a dynamical system of the repelling Écalle cylinder, i.e., a dynamical system on the space of the orbits of f, that actually corresponds to a high number of iterations of f. Studying the behaviour of this new dynamical system is the key to all the construction that will be made to prove Theorem 3.5.

Of course, while f tends to f_0, the gate gets closed, to collapse again to the single parabolic point. At the same time, the number of iterations needed to do a revolution around the two fixed points grows enormously, essentially becoming infinite for $f = f_0$.

The proof for the following proposition is mostly in [DH1] and [DH2]; the rest can be found in the appendix of [Shi].

Proposition 4.9. *Let $f_0 \in \mathcal{F}$, with $f_0'(0) = 1$ and $f_0''(0) = 1$. Then there are a neighbourhood $\mathcal{N}_0$ of f_0 and a constant $\xi_0 > 0$ (that can be the same as in Proposition 4.5) such that for any $f \in \mathcal{N}_0 \cap \mathcal{F}_1$ the assertions listed below hold.*

- *There is an analytic map* φ_f*, defined on an open set that contains at least*

$$\mathcal{Q}_f = \left\{ w \in \mathbb{C} \;\middle|\; \arg(w+\xi_0) \in \left(\frac{\pi}{3}, \frac{5\pi}{3}\right), \arg\left(w + \frac{1}{\alpha(f)} - \xi_0\right) \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right) \right\},$$

  such that

  - *for* $w, w+1 \in \operatorname{dom}\varphi_f$*, we have* $\varphi_f(w) \in \operatorname{dom} f$ *and*

$$\varphi_f \circ T(w) = f \circ \varphi_f(w);$$

  - *for* $w \in \mathcal{Q}_f$*,* $\varphi_f(w)$ *tends to* 0 *(resp.* $\sigma(f)$*) as* $\Im w$ *tends to* $+\infty$ *(resp.* $-\infty$*).*

  The function φ_f *takes both the roles of* φ_0 *and* Φ_0 *in the non perturbed case, since the distinction between attracting and repelling petals has vanished.*

- *There is an analytic map* $\mathcal{R}_f$ *defined at least on a neighbourhood of* 0 *and* ∞ *such that*

  - 0 *and* ∞ *are fixed points of* $\mathcal{R}_f$*;*
  - *no point, other than* 0 *and* ∞*, is mapped to* 0 *or to* ∞*;*
  - $\mathcal{R}_f'(0) = \exp(-2\pi i/\alpha(f))$*, hence*

$$\alpha(\mathcal{R}_f) = -\frac{1}{\alpha(f)} \pmod{\mathbb{Z}};$$

  - *let* $w, w' \in \operatorname{dom}\varphi_f$ *such that* $\mathcal{R}_f(p(w)) = p(w')$ *and*

$$\left|\arg\left(w' - w + \frac{1}{2\alpha(f)}\right)\right| < 2\pi/3; \tag{4.1}$$

    then $f^{\circ n}(\varphi_f(w)) = \varphi_f(w')$ *for some* $n \geqslant 1$*;*
  - *moreover, let* U *and* U' *be connected subsets of* $\operatorname{dom}\varphi_f$ *such that* $\mathcal{R}_f^{\circ m}(p(U)) \subseteq p(U')$ *for some* $m \geqslant 1$*; suppose also that* $\varphi_f|_U$ *and* $p|_{U'}$ *are injective and that, however taken* $w \in U$ *and* $w' \in U$*, the (4.1) is satisfied; then there is some natural number* $n > m$ *that verifies (on* $\varphi_f(U)$*):*

$$f^{\circ n} = \varphi_f \circ (p|_{u'})^{-1} \circ \mathcal{R}_f^{\circ m} \circ p \circ (\varphi_f|_U)^{-1}.$$

  The last two conditions express the fact, hinted above, that the return map $\mathcal{R}_f$ *corresponds to a certain number of iterations of the original dynamical system* f*.*

- *While* $f \to f_0$ *with* $f \in \mathcal{N}_0 \cap \mathcal{F}_1$ *it happens that*

$$\varphi_f \to \varphi_0 \qquad \text{and} \qquad \exp(2\pi i/\alpha(f))\mathcal{R}_f \to \mathcal{E}_{f_0}.$$

  Seen on the cylinder $\mathbb{C}/\mathbb{Z}$ *this means that, defining*

$$\hat{\mathcal{E}}_{f_0} := p_2^{-1} \circ \mathcal{E}_{f_0} \circ p_2 \qquad \text{and} \qquad \hat{\mathcal{R}}_f := p_2^{-1} \circ \mathcal{R}_f \circ p_2,$$

  we have

$$\hat{\mathcal{R}}_f + \frac{1}{\alpha(f)} \to \hat{\mathcal{E}}_{f_0}.$$

Using the above facts we can see the Écalle transformation $\mathcal{E}_{f_0}$ discussed in the previous section as limit of appropriate return maps: if $\{f_n\}_{n\in\mathbb{N}^*}$ is a sequence of maps of $\mathcal{F}_1$ tending to f_0 and $\{k_n\}$ integers such that $1/\alpha(f_n) - k_n \to -c$, then

$$\lim_{n\to\infty} \hat{\mathcal{R}}_{f_n} = \hat{\mathcal{E}}_{f_0} + c.$$

Remark 4.10. The Écalle transformation, being defined between two different spaces (the two Écalle cylinders), cannot, *a priori*, be interpreted as a dynamical system. However, being the limit of return maps (that are dynamical systems) as shown above, such an interpretation gets legitimate.

Note also that we have a degree of freedom on choosing the constant $c \in \mathbb{C}/\mathbb{Z}$: this corresponds to the choice of an isomorphism between the two Écalle cylinders with which compose the Écalle transformation to obtain an authentic dynamical system.

Finally, the choices for c are by no means equivalent, since the resulting dynamics exhibits completely different behaviours (for instance, the fixed point 0 of $\mathcal{E}_{f_0}$ can become attracting, repelling or indifferent). When introducing the Écalle transformation we agreed to set c in order to make 0 tangent to the identity, and this is a crucial point in the continuation of this work.

From the expression of φ_0^* of Proposition 4.5 we also get this result that will be useful later.

Proposition 4.11. *For sufficiently large $\eta > 0$ there is a neighbourhood $\mathcal{N}_1(\eta) \subseteq \mathcal{N}_0$ of f_0 such that for $f \in \mathcal{N}_1(\eta) \cap \mathcal{F}_1$ it happens that φ_f is defined and injective on the discs $D(\pm\eta)$, with $\varphi_f^*(D(\pm\eta)) \subseteq D'(\pm\eta)$.*

This immediately implies that $\left|(\varphi_f^*)'\right| \leqslant 2$ on $D(\pm\eta)$.

Proof. Let E be the (open) disc centered in $i\eta$ and with radius $|\eta|/3$, that contains $D(\eta)$. Clearly $|w| < \frac{4}{3}|\eta|$ for $w \in E$. Then write

$$\varphi_0^*(w) = w + a\log w + b + o(1) \qquad \mathcal{Q}_0 \ni w \to \infty$$

(as in Proposition 4.5) and take η large enough so that for $w \in E$ we have $|a\log w + b + o(1)| < |w|/4 < |\eta|/3$. It immediately follows that $\varphi_0^*(D(\eta)) \subset F \subset D(\eta')$, where F is the disc centered in $i\eta$ with radius $2|\eta|/3$. Moreover, since $D(\eta) \subset \mathcal{Q}_0$ for η sufficiently large, we also have that φ_0^* is injective on $D(\eta)$.

It is then easy to take a neighbourhood $\mathcal{N}_1(\eta)$ of f_0 small enough so that for $f \in \mathcal{N}_1(\eta)$ we have that $\varphi_f^*(D(\eta)) \subset D(\eta')$. Our last concern is the injectivity: take a path γ that describes once the border of a disc centered in $i\eta$ and with radius $|\eta|/6$. Since φ_0^* is injective in E and both γ and $\bar{D}(\eta)$ are compact, we can find $\varepsilon > 0$ such that for each $z \in \bar{D}(\eta)$ and $w \in \gamma$ we have $|\varphi_0^*(w) - z| > \varepsilon$.

After perhaps reducing $\mathcal{N}_1(\eta)$ we have that $|\varphi_f^* - \varphi_0^*| < \varepsilon$ on γ, for each $f \in \mathcal{N}_1(\eta)$. Then, for a fixed z, by Rouche's theorem the equation

$$\varphi_f^*(w) - z = (\varphi_f^*(w) - \varphi_0^*(w)) + (\varphi_0^*(w) - z) = 0$$

has the same number of zeros on the region bordered by γ as $\varphi_0^*(w) - z = 0$ (i.e., just one), so φ_f^* is injective on $D(\eta)$. □

4.6 Reparametrization around a parabolic point

The proofs of the following facts are mostly technical and will not be given here. They can be found in [Shi], §5.

Lemma 4.12. *Let φ_0 be the Fatou coordinate for the repelling petal of the dynamical system f_0, tangent to the identity in the origin. Then there is a one-to-one corresponance between* $\operatorname{dom}\varphi_0$ *and the set of inverse orbits for f_0 converging non-trivially to the origin: this correspondance is obtained considering, for $w \in \operatorname{dom}\varphi_0$, the inverse orbit $\{z_j\}_{j\in\mathbb{N}}$ with $z_j = \varphi_0(w-j)$.*

Additionally, if the z_j are not critical points of f_0 for $j \geqslant 1$, then $\varphi_0'(w) \neq 0$.

Lemma 4.13. *Let $f \in \mathcal{F}$ be a germ of dynamical system around the origin, there tangent to the identity and of multiplicity $n+1$. Then f has at most n immediate parabolic basins, each of which contains at least one critical value of f (unless f is a parabolic Möbius transformation) and at least one attracting petal.*

Moreover, if an immediate parabolic basin contains exactly one critical point, then it is simply connected.

Consider now f_0 tangent to the identity in the origin, with multiplicity $n+1=2$. Let $\mathcal{B}$ be its parabolic basin and $\mathcal{B}' \subset \mathcal{B}$ its immediate parabolic basin, that has just one component because of Lemma 4.13. Define $\tilde{\mathcal{B}} = \varphi_0^{-1}(\mathcal{B})$ and $\tilde{\mathcal{B}}' = \varphi_0^{-1}(\mathcal{B}')$.

From Proposition 4.5 we have that $\{\, w \in \mathbb{C} \mid |\Re w| > \eta_0 \,\} \subset \tilde{\mathcal{B}}$. But $\mathcal{B}$ contains an attracting petal, so for each $w \in \tilde{\mathcal{B}}$ there is some $n \geqslant 0$ such that $w + n = T^{\circ n}(w) \in \tilde{\mathcal{B}}'$. By connectivity, this implies that $\{\, w \in \mathbb{C} \mid |\Re w| > \eta_0 \,\} \subset \tilde{\mathcal{B}}'$.

Call then $\tilde{B}^u$ and $\tilde{B}^\ell$ respectively the upper and lower components of $\tilde{\mathcal{B}}'$ (i.e., the one that contains $\{\, w \mid \Im w > \eta_0 \,\}$ and the one that contains $\{\, w \mid \Im w < -\eta_0 \,\}$). Again by connectivity we have that $T(\tilde{B}^u) = \tilde{B}^u$ and $T(\tilde{B}^\ell) = \tilde{B}^\ell$, so it is meaningful to define the projections to $\mathbb{C}^*$, $B^u = p(\tilde{B}^u)$ and $B^\ell = p(\tilde{B}^\ell)$.

Now we will consider dynamical system in the set

$$\mathcal{F}_0 = \left\{ f \in \mathcal{F} \;\middle|\; \begin{array}{l} f'(0) = f''(0) = 1, \text{ the immediate parabolic basin of } f \\ \text{contains only one critical point} \end{array} \right\}.$$

Proposition 4.14. *Let $f_0 \in \mathcal{F}_0$ and $\tilde{B}^u$, $\tilde{B}^\ell$, B^u and B^ℓ as before. Let also be $g_0 = \mathcal{E}_{f_0}|_{B^u \cup B^\ell}$. Then:*

- *$g_0 \colon B^u \cup B^\ell \to \mathbb{C}^*$ is a branched covering of infinite degree, ramified only over one point $v \in \mathbb{C}^*$;*
- *the sets $\tilde{B}^u$, $\tilde{B}^\ell$, $B^u \cup \{0\}$ and $B^\ell \cup \{0\}$ are simply connected;*
- *B^u and B^ℓ are disjoint.*

Proposition 4.15. *After perhaps a linear scaling of coordinates, g_0 itself belongs to $\mathcal{F}_0$. Moreover, the orbit for g_0 of the critical value v is defined and it converges to 0.*

The (germ of) dynamical system g_0 is a reparametrization of f_0 and acts as a dynamical system over the space of the *orbits* of f_0. Proposition 4.15 shows that g_0 itself satisfies the hypoteses we needed for f_0, so the reparametrization con be done again and again. In the next chapter we will use this construction, while at the same time studying its peculiarities in the perturbed case, to prove Theorem 3.5.

CHAPTER 5

Dimension of Julia sets

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5.1 Preparation for building the cookie cutter

This chapter will be completely dedicated to the proof of Theorem 3.5. The ideas already presented in Chapter 4 will be furtherly analyzed, particularly we will see how the process of reparametrization around a parabolic point changes in the perturbed case. Exploiting these ideas we will be able to see some polynomials P_λ as cookie-cutter maps and then, using Lemma 2.12, we will have suitable estimates on their hyperbolic dimension.

The proof will be given only for the case $p/q = 1$ and with multiplicity $n + 1 = 2$: the general proof requires some rather more complicated notations, but uses exactly the same ideas as this one. With these hypoteses, we can assume (up to an affine coordinate change) that $f_0 \in \mathcal{F}_0$.

Let φ_0 be the map described in Proposition 4.5 and $g_0 = \mathcal{E}_{f_0}|_{U^u \cup U^\ell}$ as constructed in Section 4.6, defined on some neighbourhoods of 0 and ∞. By Proposition 4.15, $g_0 \in \mathcal{F}_0$ and g_0 has a unique critical value, v.

We can then apply again the construction of Section 4.6 to g_0, calling ψ_0 the map that plays the role of φ_0 for g_0 and $h_0 = \mathcal{E}_{g_0}|_{\tilde{U}^u \cup \tilde{U}^\ell}$ (where $\tilde{U}^u$ and $\tilde{U}^\ell$ are the U^u and U^ℓ as before for the map h_0).

Propositions 4.9 and 4.11 apply to both f_0 and g_0: we will respectively use for the objects f, φ_f, $\mathcal{R}_f$ and $\mathcal{N}_1(\eta)$ the names f, φ_f, $\mathcal{R}_f$ and $\mathcal{N}_1(f_0, \eta)$ in the case of f_0 and the names g, ψ_g, $\mathcal{R}_g$ and $\mathcal{N}_1(g_0, \eta)$ in the case of g_0.

Finally, we will keep using the notations introduced in Section 4.1.

Remark 5.1. The presence of the critical value v will cause us a bit of additional work. Some of the results of this chapter will be "duplicated", mostly with the purpose of avoiding v: at least one of the constructions will not be tainted by v, so it will give us the result we are looking for.

Inverse orbits It follows from Corollary 4.4 and Proposition 1.9 that there is an inverse orbit $\{z_j\}$ for f_0 converging non-trivially to 0, with $z_0 = 0$. By Lemma 4.12 there is a unique $\tilde{w}_0 \in \operatorname{dom} \varphi_0$ that verifies, for each $j \in \mathbb{N}$, the condition $z_j = \varphi_0(\tilde{w}_0 - j)$. Again by Corollary 4.4, $p(\tilde{w}_0) \notin \operatorname{dom} g_0$, because g_0 is defined restricted only to the horns of the Écalle cylinder.

Lemma 5.2. *Let $w_0 \in \mathbb{C}^*$. Then there exist two inverse orbits for g_0, $\{w_j\}$ and $\{w'_j\}$, with the following properties:*

- *$w_0 = w'_0$;*
- *$\{w_j\}$ and $\{w'_j\}$ are disjoint sets;*
- *$w_j, w'_j \to 0$ for $j \to \infty$;*
- *for $j \geqslant 2$, neither w_j nor w'_j are critical points for g_0;*
- *if $w_0 \notin \operatorname{dom} g_0$, then also w_1 and w'_1 are not critical points for g_0.*

Proof. Let $A := \{\, g_0^{\circ n}(v) \mid n \in \mathbb{N} \,\}$ be the forward orbit of the unique critical point v of g_0.

Clearly $g_0^{-1}(w_0)$ contains at most one periodic point for g_0, otherwise w_0 would belong to two different cycles. But then $g_0^{-1}(w_0) \cap A$ contains at most two points, since for each two points of such set at least one must be periodic. Then, being g_0 a covering of infinite degree branched only on v, we can choose $w_1, w'_1 \in g_0^{-1}(w_0) \setminus A$.

Since 0 is a parabolic point, taking a point w''_0 from the repelling petal of g_0 in 0 we can build an inverse orbit $\{w''_j\}$ for g_0, converging non-trivially to 0 and such that there is a whole neighbourhood $D' \ni w''_0$ on which $g_0^{\circ(-n)}$ converges uniformly to 0 (note that g_0 is locally invertible in 0). Moreover, $\{w''_j\}$ can be taken disjoint from A.

Let D be an open and simply connected set, disjoint from A (remember that most of the points of A are close to 0) and that contains w_1, w'_1 and w''_1. Since the orbit of v is omitted, for $j \geqslant 1$ there is an inverse branch $G_j \colon D \to \mathbb{C}^*$ of $g_0^{\circ j}$, such that $g^{\circ j} \circ G_j = Id_D$ and $G_j(w''_1) = w''_{j+1}$. Because of Proposition 1.6, the family $\{G_j\}$ is normal (it omits the points 0, ∞ and v): but then, since it converges uniformly to 0 on $D' \cap D$, it must converge to 0 on the whole D.

It the follows that $w_{j+1} := G_j(w_1)$ and $w'_{j+1} := G_j(w'_1)$ satisfy the claimed properties (for the last one, actually, it is enough to have $w_0 \neq v \in \operatorname{dom} g_0$). □

Apply this lemma to $w_0 := p(\tilde{w}_0)$ and let $\{w_j\}$ and $\{w'_j\}$ be as above. Using again Lemma 4.12 we have that there are $\tilde{\zeta}_0, \tilde{\zeta}'_0 \in \operatorname{dom} \psi_0$ such that $w_j = \psi_0(\tilde{\zeta}_j)$ and $w'_j = \psi_0(\tilde{\zeta}'_j)$ for each j. Aditionally, the w_j and w'_j are not critical for g_0, so $\psi'_0(\tilde{\zeta}_0)$ and $\psi'_0(\tilde{\zeta}'_0)$ are not zero.

Perturbation of the second Écalle cylinder We now start perturbing the dynamical system, starting from h_0: the following lemma gives us the key to trace how the orbits change when f_0 (thus g_0 and h_0) are slightly modified.

Since the dynamics expressed by this lemma are more easily described on the cylinder $\mathbb{C}/\mathbb{Z}$ instead of $\mathbb{C}^*$, we define $\hat{\zeta}_0 := p_1(\tilde{\zeta}_0)$ and $\hat{\zeta}_0' := p_1(\tilde{\zeta}_0')$.

Lemma 5.3. *Let $b > 0$. Then there are a neighbourhood $\mathcal{N}_2$ of h_0 and two disjoint discs $W, W' \subset \mathbb{C}/\mathbb{Z}$ that respectively contain $\hat{\zeta}_0$ and $\hat{\zeta}_0'$ that, for some positive constants C_0 and C_1 satisfy the following property: let $h_1 \in \mathcal{N}_2$ and $\beta \in \mathbb{C}/\mathbb{Z}$ with $|\Im\beta| \leqslant b$; let also*

$$\hat{h}(\zeta) := p_2^{-1} \circ h_1 \circ p_2(\zeta) - \beta \qquad \text{for } \zeta \in p_2^{-1}(\operatorname{dom} h_1) \subset \mathbb{C}/\mathbb{Z};$$

then there is a sequence $\{W_j\}$ of disjoint topological discs of $\mathbb{C}/\mathbb{Z}$ such that

- *W_0 is W or W';*
- *for each j, the map $\hat{h}|_{W_{j+1}}$ is defined and it is a bijection onto W_j;*
- *the discs W_j escape from any compact of $\mathbb{C}/\mathbb{Z}$ as $j \to \infty$ (more precisely, they tend to the upper or to the lower end of $\mathbb{C}/\mathbb{Z}$);*
- $\operatorname{diam} W_j < \frac{1}{2}$ *for each j;*
- *considering on the cylinder $\mathbb{C}/\mathbb{Z}$ the metric d induced by its covering with the plane $\mathbb{C}$*

$$\inf_{\substack{x \in W_j,\\ y \in W_{j+1}}} d(x, y) =: d(W_j, W_{j+1}) < C_0;$$

- $\left|(\hat{h}^{\circ j}|_{W_j})'\right| < C_1$ *for each j.*

Proof. Consider initially the case of the map $h := e^{2\pi i\beta}h_0$. Since h_0 has been constructed the same way as g_0, the sets $\tilde{B}^u$, $\tilde{B}^\ell$, B^u and B^ℓ can be defined for h_0 as well. Throughout this proof we will consider them for h_0 instead of g_0.

From Proposition 4.14 we know that at most one of B^u and B^ℓ contains the unique critical value v: take the other, for instance B^u. Then, by simple connectivity, the local inverse of h near 0 can be extended to $B^u \cup \{0\}$, becoming $H\colon B^u \cup \{0\} \to B^u \cup \{0\}$, with $H(0) = 0$ and $h \circ H = Id$. Moreover, because of Schwarz' lemma, $|H'(0)| < 1$ (since if it were $= 1$, then H would be a rotation; clearly it is not, for instance because it is a covering of infinite degree).

Since 0 is an attracting point for H, by the Kœnigs linearization theorem (Thorem 8.2 in [Mil]), there exist a local linearizing coordinate $L(z)$, such that $L(0) = 0$, $L'(0) = 1$ and

$$L \circ H(z) = H'(0) \cdot L(z)$$

in a neighbourhood of 0.

Moreover, by Theorem 5.2 in [Mil], we also know that for each $z \in B^u \cup \{0\}$ the orbit $H^{\circ n}(z)$ converges to 0 for $n \to \infty$.

Now pass H to the $\mathbb{C}/\mathbb{Z}$ cylinder, taking $\hat{H} = p_2^{-1} \circ H \circ p_2$. The linearizing coordinate changes as it follows. Denoting as $Y := \{\, z \in \mathbb{C}/\mathbb{Z} \mid \Im z > y_0 \,\}$ as the upper part of the cylinder, for some $y_0 > 0$, there are an analytic function $\hat{L} = p_2^{-1} \circ L \circ p_2 \colon Y \to \mathbb{C}/\mathbb{Z}$ and a constant $C'' > 0$ such that $\hat{H}$ is defined on Y, for $z \in Y$ we have $\Im(\hat{H}(z) - z) < C''$ and

$$\hat{L} \circ \hat{H} = \hat{L} + a$$

where $a = \frac{1}{2\pi i} \log H'(0)$, with $\Im a > 0$.

Now take $\hat{\zeta}_0$ and $\hat{\zeta}_0'$: at least one of them it is not a critical value of $\hat{h}$, say $\hat{\zeta}_0$. Then pick $\hat{\zeta}_1 \in \hat{h}^{-1}(\hat{\zeta}_0) \cap B^u$ and define $\hat{\zeta}_{j+1} := H^{\circ j}(\hat{\zeta}_1)$ for $j > 1$. Since such points tend to the upper end of the cylinder, there is $j_0 \geqslant 1$ such that $\hat{\zeta}_{j_0} \in Y$. The existence of the linearizing coordinate becomes important here, because it enables us to prove the needed estimates for a finite number j_0 of discs W_j (before they enter the domain Y) and then extending the estimates to all the others, exploiting the fact that we know how the dynamical system work near the 0.

We can take a small disc W_0 that contains $\hat{\zeta}_0$ and its inverse images $W_1, \ldots, W_{j_0}$ through $\hat{H}$ such that for $j = 1, \ldots, j_0$ it happens that $\hat{\zeta}_j \in W_j$, the closures $\bar{W}_j$ are disjoint and contained in $B^u \setminus Y$ with the exception of $\bar{W}_{j_0}$ that is in Y.

The properties of the sequence $\hat{\zeta}_j$ and of the discs $W_0, \ldots, W_{j_0}$ are stable under perturbation of h; i.e., they are still valid (and with uniform constants) taking $h = e^{2\pi i \beta'} h_1$, where β' is near β and h_1 is near h_0.

By compactness of $\{\, \beta \in \mathbb{C}/\mathbb{Z} \mid |\Im \beta| \leqslant b \,\}$ we can find a neighbourhood $\mathcal{N}_2$ of h_0, two disjoint discs W and W' containing respectively $\hat{\zeta}_0$ and $\hat{\zeta}_0'$ such that the afore mentioned properties are verified with uniform constants for all $h_1 \in \mathcal{N}_2$ and β with $|\Im \beta| \leqslant b$, taking $W_0 = W$ or $W_0 = W'$.

Since we are still dealing with a finite number of discs W_j, it easy to have uniform estimates on their dimension and distance. Now we define $W_j := \hat{H}^{\circ j - j_0}(W_{j_0})$ for $j > j_0$: the metric properties of $\hat{H}$ imply that such estimates can be extended uniformly to the whole sequence of discs. The estimate on the derivative easily follows from the others (using Schwarz' lemma). □

Projection to the dynamical space of h_0 We are now interested into tracing the preimages of the discs W_j just created along the projections to the Écalle cylinders of the two dynamical systems f_0 and g_0.

Lemma 5.4. *There is a constant $\gamma > 0$ (depending only on C_0) such that for large $\eta > 0$, taken $\eta' = e^{2\pi\eta}$ or $\eta' = -e^{2\pi\eta}$ and a perturbation $f \in \mathcal{N}_1(f_0, \eta) \cap \mathcal{F}_1$ such that $g \in \mathcal{N}_1(g_0, \eta') \cap \mathcal{F}_1$, there are disjoint topological discs $U_1, \ldots, U_N$ of $\mathbb{C}$ with the following properties:*

- *$N \geqslant \gamma \eta \exp(4\pi\eta)$;*
- *$\bar{U}_i \subset \varphi_f(D(\eta)) \subset D'(\eta)$ for each i;*
- *$V_i := p \circ (\varphi_f|_{D(\eta)})^{-1}(U_i) \subset \psi_g(D(\eta)) \subset D'(\eta')$ for each i;*
- *for each i there is some $j(i)$ such that $W_{j(i)} = p_1^{-1} \circ (\psi_g|_{D(\eta')})^{-1}(V_i)$;*

- *the following estimates on the derivatives hold:*

$$\left|(\tau_0|_{U_i})'\right| < \frac{9}{4}\eta^2, \qquad \left|(\pi^*|_{\tilde{V}_i})'\right| < 3\pi e^{2\pi\eta},$$

where $\tilde{V}_i := (\varphi_f|_{D(\eta)})^{-1}(U_i)$.

Proof. Let us suppose that the discs W_j tend to the upper end on the cylinder $\mathbb{C}/\mathbb{Z}$ and take $\eta' = e^{2\pi\eta}$. The other case is similar, using $\eta' = -e^{2\pi\eta}$ instead.

Because of the properties of uniform spacing of the W_j, there is a constant $\gamma' > 0$ (depending only on C_0) such that a disc $D(\eta')$ contains entirely the closure of at least $\gamma'(\eta')^2$ connected components of the set $p_1^{-1}(\bigcup_j W_j)$. It follows from Proposition 4.11 that ψ_g^* maps those components injectively into $D'(\eta')$.

Each component of the inverse image of $D'(\eta')$ via p^* stays in a "box"

$$B_n := (n,\, n+1) + i\left(\eta + \frac{1}{2\pi}\log\frac{1}{2},\, \eta + \frac{1}{2\pi}\log\frac{3}{2}\right)$$

for some $n \in \mathbb{Z}$. As before, $D(\eta)$ contains at least $\eta/3$ of such components and φ_f^* maps $D(\eta)$ injectively in $D'(\eta)$.

Setting then $\gamma := \gamma'/3$ and

$$\begin{aligned}\Phi &:= p_1 \circ (\psi_g^*|_{D(\eta')})^{-1} \circ p^* \circ (\varphi_f^*|_{D(\eta)})^{-1} \circ \tau_0 \\ &= p_1 \circ (\psi_g|_{D(\eta')})^{-1} \circ p \circ (\varphi_f|_{D(\eta)})^{-1}\end{aligned}$$

we have that there are at least $N \geqslant \gamma\eta(\eta')^2$ components U_i of the inverse image through Φ of the discs W_j that satisfy the thesis of the theorem. In particular, the estimates on the derivatives of τ_0 and p^* descend respectively from the facts that $U_i \subset D'(\eta)$ and $\tilde{V}_i \subset B_n$ for some n. □

Return to the dynamical space of f_0 To complement the construction made in the previous paragraph we have to build the inverse of the projection Φ on the set W_0.

Consider again the succession $\{z_j\}$ built above: since it converges non-trivially to 0, it is definitively contained in the attracting petal of 0; then, since f_0 has at most one critical point in its immediate parabolic basin, there is $k \geqslant 0$ such that for $j > k$ the point z_j is not critical for f_0. Define $p^{(-1)}$ the local inverse of p near w_0 such that $p^{(-1)}(w_0) = \tilde{w}_0 - k$ and $p_1^{(-1)}$ the local inverse of p_1 such that $p_1^{(-1)}(\zeta_0) = \tilde{\zeta}_0$ and $p_1^{(-1)}(\zeta_0') = \tilde{\zeta}_0'$. It then follows from Lemma 4.12 that $\varphi_0'(\tilde{w}_0 - k) \neq 0$ (we also already proved that $\psi_0'(\tilde{\zeta}_0) \neq 0$ and $\psi_0'(\tilde{\zeta}_0') \neq 0$).

Thus, after perhaps making W and W' smaller (that doesn't void anything of what was discussed above), there are a small neighbourhood U of z_k, neighbourhoods $\mathcal{N}_3(f_0)$ and $\mathcal{N}_3(g_0)$ respectively of f_0 and g_0 and a positive constant C_2 such that, for $f \in \mathcal{N}_3(f_0) \cap \mathcal{F}_1$ and $g \in \mathcal{N}_3(g_0) \cap \mathcal{F}_1$:

- the function $\Phi^{(-1)} := \varphi_f \circ p^{(-1)} \circ \psi_g \circ p_1^{(-1)}$ is defined in injective on W and W'; note that $\Phi^{(-1)}$ is formally the inverse of Φ, although defined on a different domain;
- $U \subset \Phi^{(-1)}(W), \Phi^{(-1)}(W')$;
- $\left|(\Phi^{(-1)})'\right| < C_2$ on $W \cup W'$.

Last iterations Finally, $f_0^{\circ k}$ (that is a rational and non constant, thus open function) sends U onto a neighbourhood of 0. Then we can take an open subset $\mathcal{U}' \subset U$ (not necessarily containing z_k) such that $f_0^{\circ k}$ is a diffeomorphism from $\mathcal{U}'$ onto $\mathcal{U} := \tau_0(D'(\eta))$, for large enough $\eta > 0$ (since the set $\mathcal{U}$ tends uniformly to zero while $\eta \to \infty$).

To estimate the derivative, we have to consider the inverse function $f_0^{\circ(-k)}$: its derivative is a rational function, that gets evaluated on $\mathcal{U}$; this means that the derivative of $f_0^{\circ k}$ on $\mathcal{U}'$ grows at most polynomially with η.

As usual, perhaps after changing slightly $\mathcal{U}$, the same properties are retained when substituting f_0 with f belonging to a suitable neighbourhood $\mathcal{N}_4(f_0, \eta)$ of f_0 (of course $\mathcal{U}'$ depends on f).

5.2 Building the cookie cutter

In this final section we use the elements already developed in the previous one to finally prove Theorem 3.5.

Proof of Theorem 3.5 (in the case $p/q = 1$). Consider (up to an affine coordinates change) $f_0 \in \mathcal{F}_0$. If f is close to f_0, then it can be conjugated to a function in $\mathcal{F}$ with a translation, so we can just prove the theorem for $f \in \mathcal{F}$. Suppose also that

$$\alpha(f) = \frac{1}{a_1 - \frac{1}{a_2+\beta}}$$

for integers $a_1, a_2 > 0$ (the other choices of signs are completely analogous; the only difference is wether we have to consider one cylinder's end or the other).

If a_1 and a_2 are large enough and f is close enough to f_0, then Proposition 4.9 applies and the objects φ_f and $\mathcal{R}_f$ are defined. Calling $g := \mathcal{R}_f$, we have that $\alpha(g) = \frac{1}{a_2+\beta}$ and that $e^{2\pi i/\alpha(f)}g$ is close to $g_0 := \mathcal{E}_{f_0}$. Then Proposition 4.9 can be applied again, giving $h := \mathcal{R}_g$ and $h_0 := \mathcal{E}_{g_0}$, with $\alpha(h) = -\beta$ and $h_1 := e^{2\pi i/\alpha(g)}h = e^{2\pi i\beta}h$ close to h_0.

Pick now a large $\eta > 0$, $\eta' = e^{2\pi\eta}$ and f such that

$$\begin{aligned} f &\in \mathcal{N}_1(f_0, \eta) \cap \mathcal{N}_3(f_0) \cap \mathcal{N}_4(f_0, \eta), \\ g &\in \mathcal{N}_1(g_0, \eta') \cap \mathcal{N}_3(g_0), \\ h_1 &\in \mathcal{N}_2. \end{aligned}$$

The hypoteses of Lemmas 5.3 and 5.4 are satisfied, so there are objects W_j, U_i, V_i and $\tilde{V}_i$ as previously discussed.

Let also $U, \mathcal{U}, \mathcal{U}'$, $p^{(-1)}$ and $p_1^{(-1)}$ be as described at the end of the previous section and

$$U_i^* := \tau_0^{-1}(U_i), \qquad V_i^* := \tau_0^{-1}(V_i),$$
$$V := \psi_g \circ p_1^{(-1)}(W_0), \qquad U' := \varphi_f \circ p^{(-1)}(V) \subseteq U;$$

we obtain this composition:

$$
\begin{array}{ccccccccc}
U_i & \xrightarrow{\tau_0} & U_i^* & \xrightarrow{(\varphi_f^*|_{D(\eta)})^{-1}} & \tilde{V}_i & \xrightarrow{\pi^*} & V_i^* & \xrightarrow{\pi_1 \circ (\psi_g^*|_{D(\eta')})^{-1}} & W_{j(i)} \\
 & & & & & & & & \big\downarrow \hat{h}^{\circ j(i)} \\
 & & f^{\circ k}(U') & \xleftarrow{f^{\circ k}} & U' & \xleftarrow{\varphi_f \circ \pi^{(-1)}} & V & \xleftarrow{\psi_g \circ \pi_1^{(-1)}} & W_0.
\end{array}
$$

For each i, all these maps are bijections, except for the last $f^{\circ k}$, which may fail to be injective. We have to reduce, for the last time, U' to $\mathcal{U}'$ (so that $f^{\circ k}(\mathcal{U}') = \mathcal{U}$), than trace back the inverse images of $\mathcal{U}$ and $\mathcal{U}'$ along all the maps. In particular, the U_i are reduced to $\mathcal{U}_i \subseteq U_i$ such that $\bar{\mathcal{U}}_i \subset \mathcal{U}$. Call $F_i \colon \mathcal{U}_i \to \mathcal{U}$ the composition of the map above: now it is bijective.

To show that f is actually a cookie cutter, we now must see that, for each i, F_i is actually an iterate of f: to do that the last unused property of $\mathcal{R}_f$ in Proposition 4.9 will be employed. For fixed η and η', if a_2 is large enough the condition

$$\left| \arg\left(z' - z + \frac{1}{2\alpha(g)} \right) \right| = \left| \arg\left(z' - z - \frac{a_2 + \beta}{2} \right) \right| < \frac{2\pi}{3}$$

is satisfied for $z \in D(\eta')$ and $z' \in p_1^{(-1)}(W \cup W')$. Thus we can write

$$
\begin{aligned}
g^{\circ m_i} &= \psi_g \circ p^{(-1)} \circ h^{\circ j(i)} \circ p \circ (\psi_g|_{D(\eta')})^{-1} \\
&= \psi_g \circ p_1^{(-1)} \circ \hat{h}^{\circ j(i)} \circ p_1 \circ (\psi_g|_{D(\eta')})^{-1}
\end{aligned}
$$

on V_i for some numbers m_i. For exactly the same reason, for a_1 sufficiently large we have

$$f^{\circ n_i'} = \varphi_f \circ p^{(-1)} \circ g^{\circ m_i} \circ p \circ (\varphi_f|_{D(\eta)})^{-1}$$

on $\mathcal{U}_i$, for some numbers n_i.

Putting together the last formulae we have that there are numbers $n_i := n_i' + k$ such that, for each i, we have $F_i \equiv f^{\circ n_i}|_{\mathcal{U}_i}$: we have proved that f is a cookie cutter on the sets $\mathcal{U}_i$.

To conclude the proof we just have to apply Lemma 2.12. Since Lemma 5.4 we know that

$$N \geqslant \gamma\eta \exp(4\pi\eta)$$

and from the estimates arising from Proposition 4.11, Lemmas 5.3 and 5.4 and the conclusion of Section 5.1 it follows that, for each i,

$$|F_i'| \leqslant C\mu(\eta)\exp(2\pi\eta),$$

where C is a constant and μ a real rational function (both independent from i).

It finally follows that

$$\text{hyp-dim}\, f \geqslant \frac{\log N}{\log(\max_i \sup_{U_i} |(f^{\circ n_i})'|)} \geqslant \frac{\log\gamma + \log\eta + 4\pi\eta}{\log C + \log\mu(\eta) + 2\pi\eta},$$

that tends to 2 as $\eta \to \infty$. The theorem is then proved. □

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